

DUALITY AND (q -)MULTIPLE ZETA VALUES

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ABSTRACT. Following Bachmann's recent work on bi-brackets and multiple Eisenstein series, Zudilin introduced the notion of multiple q -zeta brackets, which provides a q -analog of multiple zeta values possessing both shuffle as well as quasi-shuffle relations. The corresponding products are related in terms of duality. In this work we study Zudilin's duality construction in the context of classical multiple zeta values as well as various q -analogs of multiple zeta values. Regarding the former we identify the derivation relation of order two with a Hoffman–Ohno type relation. Then we describe relations between the Ohno–Okuda–Zudilin q -multiple zeta values and the Schlesinger–Zudilin q -multiple zeta values.

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1. INTRODUCTION

Multiple zeta values (MZVs) are nested sums of depth $n \in \mathbb{N}$ and weight $\sum_{i=1}^n k_i > n$, for positive integers $k_1 > 1, k_i > 0, i = 2, \dots, n$

$$(1) \quad \zeta(k_1, \dots, k_n) := \sum_{m_1 > \dots > m_n > 0} \frac{1}{m_1^{k_1} \dots m_n^{k_n}}.$$

They arise in various contexts, e.g., number theory, algebraic geometry, algebra, as well as knot theory. The origins of the modern systematic treatment of MZVs can be traced to [7, 26, 8]. Moreover, MZVs and their generalisations, i.e., multiple polylogarithms, play an important role in quantum field theory [3].

It is well-known that the nested sums in (1) can also be written as iterated Chen integrals, which induces *shuffle relations* among MZVs thanks to integration by parts. Multiplying MZVs directly, on the other hand, yields *quasi-shuffle relations*. Comparison of the different products results in a large collection of so-called *double shuffle relations* among MZVs [11, 29, 9, 25]. Amending a regularization procedure with respect to the formally defined single zeta value $t := \zeta(1)$ leads to *regularized double shuffle relations*, which conjecturally give all linear relations among MZVs [13].

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A particular change of variables in the context of the integral representation of MZVs gives way to a peculiar set of relations subsumed under the notion of *duality* for MZVs. As an example we state the identity

$$\zeta(5, 1) = \zeta(3, 1, 1, 1).$$

A precise understanding of the mathematical relation between the notion of duality and the aforementioned double shuffle structure is part of a class of important open problems in the theory of MZVs. See [13] for details.

Generalizations of the real-valued nested sums in (1) to power series in $\mathbb{Q}[[q]]$ are commonly known as q -analogues of MZVs. A particular example of such q -MZVs is due to Bradley and Zhao [2, 27], who extended a q -analog of the Riemann zeta function introduced by Kaneko et al. [15]. More recently, several q -analogues of MZVs were shown to satisfy – regularized – double shuffle relations [5, 22, 21, 23]. The notion of duality in the context of those q -MZVs is more complicated. See [28] for details. Inspired by Bachmann’s intriguing work [1], Zudilin presents in [30] a particular model called *multiple q -zeta brackets*, which possesses a natural quasi-shuffle product. After multiplying Zudilin’s multiple q -zeta brackets with a certain positive integer power of $1 - q$ one obtains ordinary MZVs in the classical limit $q \rightarrow 1$. The key result in [30] is an algebraic duality-type construction that permits to deduce a shuffle product for multiple q -zeta brackets from the quasi-shuffle product of the model. This fact was shown by Bachmann to hold in depth one and two [1]. In a nutshell, this construction works as follows: Let (\mathcal{A}, m) be an algebra, and $\zeta : (\mathcal{A}, m) \rightarrow k$ is a multiplicative linear map from \mathcal{A} into the ring k . The map $\tau : \mathcal{A} \rightarrow \mathcal{A}$ is a particular involution, i.e., $\tau \circ \tau = \text{Id}$. Both maps ζ and τ are compatible in the sense that $\zeta \circ \tau = \zeta$. Then the *dual product* $m_{\square} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ corresponding to the original product m on \mathcal{A} is defined by

$$(2) \quad m_{\square} := \tau \circ m \circ (\tau \otimes \tau).$$

It turns out that $\zeta : (\mathcal{A}, m_{\square}) \rightarrow k$ is again a multiplicative linear map into k . In the case of multiple q -zeta brackets Zudilin proved that if (\mathcal{A}, m) is the quasi-shuffle algebra induced by a sum representation of his model, then the dual product m_{\square} yields a shuffle product on multiple q -zeta brackets. The reason for calling it shuffle product comes from the surprising fact that, in the limit $q \uparrow 1$, the product m_{\square} reduces to the usual shuffle product of classical MZVs.

The aim of our work is to explore Zudilin’s construction in the context of other q -analogues of MZVs as well as classical MZVs. To this end we start by extending (2) to general Hopf algebras (Proposition 3.1). The result of this exercise is then applied in different settings. Firstly, we consider classical MZVs. Note that Zudilin already remarked that in this case the product dual to the quasi-shuffle product does not coincide with the shuffle product for MZVs. However, we show how to relate Zudilin’s dual product via the construction of a Hoffman– Ohno relation to a derivation relation for MZVs (Theorem 4.4).

Secondly, we study Zudilin’s duality construction in the context of the Schlesinger–Zudilin (SZ) model of q -MZVs (see (7) below) using a notion of duality given by Zhao in [28]. The specific property of the SZ model is that its quasi-shuffle product is of the same form as that of usual MZVs. Furthermore, we note that a shuffle product for (7) has already been constructed by the third author in [21] using Rota–Baxter operators, and the resulting double shuffle picture for those q -MZVs is well-understood. Our study shows that, analogous to the case of multiple q -zeta brackets, the quasi-shuffle product and the shuffle product for Schlesinger–Zudilin q -MZVs are related through duality (Theorem 5.4).

Our main application of Zudilin’s duality construction takes place in the context of the Ohno–Okuda–Zudilin (OOZ) q -model for MZVs (see (10) below). Both the quasi-shuffle-like product as well as the shuffle product were explored by Castillo Medina et al. in [4, 5]. Note that the former is more involved than that of usual MZVs (see Subsection 6.2). We also remark that this model can be defined for any integer argument since the q -parameter provides an appropriate regularization [6]. Whereas for classical MZVs as well as for Schlesinger–Zudilin q -MZVs the duality relation is of the form $\zeta \circ \tau = \zeta$, the situation in the case of OOOZ q -MZVs is different. It turns out that the quasi-shuffle-like product for OOOZ q -MZVs is related by duality to the shuffle product for SZ q -multiple zeta star values (q -MZSVs) (see Theorem 5.12). The algebraic structure of the shuffle product in the OOOZ q -model in turn is induced by the quasi-shuffle product of the SZ q -MZSVs. Applying the duality construction leads again to the shuffle product presented in [5] (see Theorem 5.12 below) as long as we restrict ourselves to non-negative integer arguments. In fact, at this stage we are unable to extend the duality construction relating OOOZ q -MZVs and SZ q -MZSVs to negative or mixed sign integer arguments.

The OOOZ model is also of interest regarding renormalization of MZVs, in a way that is compatible with the shuffle product. Indeed, by applying a theorem due to Connes and Kreimer, we renormalize MZVs in [6] preserving both the shuffle relations as well as meromorphic continuation. In the present paper we construct an infinitesimal bialgebra, which provides a bialgebra that was originally defined in the context of arbitrary integer arguments. If we

restrict this construction to non-negative arguments in the context of the quasi-shuffle product, then the coproduct of convergent words coincides with the coproduct obtained from the duality construction. As a result the left-coideal of convergent words is transferred to a right-coideal (Theorem 6.3).

The paper is organized as follows. In Section 2, we introduce algebras and subalgebras of noncommutative words over various alphabets which are relevant for classical multiple zeta values and several of their q -analogues. In Section 3 we detail the structure maps (multiplication, comultiplication, unit, counit, antipode) of a Hopf algebra, which is obtained by transferring a given Hopf algebra by means of a vector space isomorphism T . As a first application we address in Section 4 the case of classical MZVs. Section 5 is devoted to the study of Hopf algebras transferred by an involution, in the context of particular q -analogues of MZVs. Two different involutions τ and $\tilde{\tau}$ are at play, leading to two distinct families of duality relations (Theorem 5.9), which we express in the Ohno–Okuda–Zudilin model [19]. The family associated with τ leads to the classical family of duality relations in the limit $q \rightarrow 1$, whereas the second family associated with $\tilde{\tau}$, derived from [28, Theo. 8.3], does not seem to have any classical counterpart.

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2. WORD-ALGEBRAIC BACKGROUND

The polynomial algebra in two non-commutative variables p, y is denoted by $\mathfrak{H} := \mathbb{Q}\langle p, y \rangle$. The unit of $\mathbb{Q}\langle p, y \rangle$ is denoted by $\mathbf{1}$. The subalgebra of words not ending in the letter p is denoted by $\mathfrak{H}^1 := \mathbb{Q}\mathbf{1} \oplus \mathbb{Q}\langle p, y \rangle y$ and the subalgebra of words not beginning with the letter y is denoted by $\mathfrak{H}^{(-1)} := \mathbb{Q}\mathbf{1} \oplus p\mathbb{Q}\langle p, y \rangle$. The subalgebra $\mathfrak{H}^1 \cap \mathfrak{H}^{(-1)} = \mathbb{Q}\mathbf{1} \oplus p\mathbb{Q}\langle p, y \rangle y$ of words not beginning with the letter y and not ending in p is denoted by \mathfrak{H}^0 . The subalgebra \mathfrak{H}^1 can be expressed as $\mathbb{Q}\langle z_k : k \in \mathbb{N}_0 \rangle$ where z_k stands for the block $p^k y$. The subalgebra \mathfrak{H}^0 is linearly spanned by the words $z_{k_1} \cdots z_{k_n}$ with $k_1 \geq 1$ and $k_j \in \mathbb{N}_0$ for $j \geq 2$.

The polynomial algebra in two non-commutative variables x_0, x_1 will be denoted by $\mathfrak{h} := \mathbb{Q}\langle x_0, x_1 \rangle$. The unique algebra morphism $\Phi : \mathfrak{H} \rightarrow \mathfrak{h}$ such that $\Phi(p) := x_0$ and $\Phi(y) := x_1$ is obviously an isomorphism. We define $\mathfrak{h}^1, \mathfrak{h}^{-1}$ and \mathfrak{h}^0 similarly as their capitalized counterparts, hence $\mathfrak{h}^1 = \Phi(\mathfrak{H}^1)$ and so on. The subalgebra \mathfrak{h}^1 can be expressed as $\mathbb{Q}\langle z_k : k \in \mathbb{N} \rangle$ where z_k stands for the block $x_0^{k-1} x_1$. The subalgebra \mathfrak{h}^0 is linearly spanned by the words $z_{k_1} \cdots z_{k_n}$ with $k_1 \geq 2$ and $k_j \in \mathbb{N}$ for $j \geq 2$, called *convergent words* in view of (1). The unique algebra morphism $J : \mathfrak{h} \rightarrow \mathfrak{H}$ such that $J(x_0) := p$ and $J(x_1) := py$ is injective. Hence we will consider \mathfrak{h} as a subalgebra of \mathfrak{H} by identifying x_0 with p and x_1 with py . This convention renders both definitions of the letters $z_k, k \in \mathbb{N}_0$, consistent. Let us remark for later use the inclusions:

$$\mathfrak{h} \subset \mathfrak{H}^{(-1)}, \quad \mathfrak{h}^1 \subset \mathfrak{H}^0, \quad \mathfrak{h}^0 \subset \mathfrak{H}^0.$$

The unique anti-automorphism $\tilde{\tau}$ of \mathfrak{H} such that $\tilde{\tau}(p) := y$ and $\tilde{\tau}(y) := p$ is involutive. It exchanges \mathfrak{H}^1 and $\mathfrak{H}^{(-1)}$, and preserves \mathfrak{H}^0 . Accordingly, the unique anti-automorphism τ of \mathfrak{h} such that $\tau(x_0) := x_1$ and $\tau(x_1) := x_0$ is involutive, exchanges \mathfrak{h}^1 and $\mathfrak{h}^{(-1)}$, and preserves \mathfrak{h}^0 . We obviously have $\tilde{\tau} = \Phi^{-1} \circ \tau \circ \Phi$. The algebra \mathfrak{H} and all subalgebras defined above are graded by the *weight*, defined on the generators by $\text{wt}(p) := 1$ and $\text{wt}(y) := 0$. Let us remark at this stage that τ respects the weight whereas $\tilde{\tau}$ does not, and that $\tilde{\tau}$ does not preserve convergent words, i.e., $\tilde{\tau}(\mathfrak{h}^0) \not\subset \mathfrak{h}^0$.

3. TRANSFERRED HOPF ALGEBRA STRUCTURES

Before starting we briefly recall a few co- and Hopf algebra notions [16]. Let C be a coalgebra with coproduct Δ_C and counit ϵ_C . A subspace $J \subset C$ is called a left (right) *coideal* if $\Delta_C(J) \subset C \otimes J$ ($\Delta_C(J) \subset J \otimes C$). A right (left) *comodule* over C is a k -vector space M together with a linear map $\phi : M \rightarrow M \otimes C$, such that $(\text{Id}_M \otimes \Delta_C) \circ \phi = (\phi \otimes \text{Id}_C) \circ \phi$ and $(\text{Id}_M \otimes \epsilon_C) \circ \phi = \text{Id}_M$ (analogously for left comodules). Let $(\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ be a Hopf algebra. The flip map is denoted $s : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$, and defined by $s(a \otimes b) := b \otimes a$. Then the *opposite Hopf algebra* $(\mathcal{H}, m_{op}, \eta, \Delta_{op}, \epsilon, S)$ is given by $m_{op} := m \circ s$ and $\Delta_{op} := s \circ \Delta$. Further, for the coproduct $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$

and counit $\epsilon : \mathcal{H} \rightarrow k$ we introduce the notation $\Delta_{\mathcal{H} \otimes \mathcal{H}} := (\text{Id} \otimes s \otimes \text{Id})(\Delta \otimes \Delta)$ and $\epsilon_{\mathcal{H} \otimes \mathcal{H}} := \epsilon \otimes \epsilon$. It will be convenient to apply Sweedler's notation

$$\Delta(x) = \sum_{(x)} x_1 \otimes x_2.$$

Let k be a field of characteristic zero and $(\mathcal{R}, m_{\mathcal{R}}, \eta_{\mathcal{R}})$ an unital associative k -algebra. Our starting point is a Hopf algebra $(\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ over k equipped with a \mathcal{R} -valued character, i.e., a multiplicative linear map $\xi : \mathcal{H} \rightarrow \mathcal{R}$, $\xi \circ m = m_{\mathcal{R}} \circ (\xi \otimes \xi)$. Subsequently we transfer the whole structure on another k -vector space \mathcal{H}_{\square} by means of a k -linear isomorphism $T : \mathcal{H}_{\square} \rightarrow \mathcal{H}$. We call \mathcal{H}_{\square} the *transferred Hopf algebra* (*T-Hopf algebra*). This principle has been used by Zudilin [30].

Proposition 3.1 (*T-Hopf algebra*). *Let $(\mathcal{R}, m_{\mathcal{R}}, \eta_{\mathcal{R}})$ be an unital associative k -algebra and $(\mathcal{H}, m, \eta, \Delta, \epsilon, S)$ is a Hopf algebra with character $\xi : \mathcal{H} \rightarrow \mathcal{R}$. We assume the existence of a k -linear isomorphism $T : \mathcal{H}_{\square} \rightarrow \mathcal{H}$ between the k -vector spaces \mathcal{H}_{\square} and \mathcal{H} . Then $(\mathcal{H}_{\square}, m_{\square}, \eta_{\square}, \Delta_{\square}, \epsilon_{\square}, S_{\square})$ is a Hopf algebra with character $\xi_{\square} : \mathcal{H}_{\square} \rightarrow \mathcal{R}$ with*

- *multiplication: $m_{\square} : \mathcal{H}_{\square} \otimes \mathcal{H}_{\square} \rightarrow \mathcal{H}_{\square}$, $m_{\square} := T^{-1} \circ m \circ (T \otimes T)$,*

$$\begin{array}{ccc} \mathcal{H}_{\square} \otimes \mathcal{H}_{\square} & \xrightarrow{m_{\square}} & \mathcal{H}_{\square} \\ T \otimes T \downarrow & & \uparrow T^{-1} \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{m} & \mathcal{H} \end{array}$$

- *unit: $\eta_{\square} : k \rightarrow \mathcal{H}_{\square}$, $\eta_{\square} := T^{-1} \circ \eta$,*

$$\begin{array}{ccc} & k & \\ \eta \swarrow & & \searrow \eta_{\square} \\ \mathcal{H} & \xrightarrow{T^{-1}} & \mathcal{H}_{\square} \end{array}$$

- *coproduct: $\Delta_{\square} : \mathcal{H}_{\square} \rightarrow \mathcal{H}_{\square} \otimes \mathcal{H}_{\square}$, $\Delta_{\square} := (T^{-1} \otimes T^{-1}) \circ \Delta \circ T$,*

$$\begin{array}{ccc} \mathcal{H}_{\square} & \xrightarrow{\Delta_{\square}} & \mathcal{H}_{\square} \otimes \mathcal{H}_{\square} \\ T \downarrow & & \uparrow T^{-1} \otimes T^{-1} \\ \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H} \otimes \mathcal{H} \end{array}$$

- *counit: $\epsilon_{\square} : \mathcal{H}_{\square} \rightarrow k$, $\epsilon_{\square} := \epsilon \circ T$,*

$$\begin{array}{ccc} & \mathcal{H}_{\square} & \\ T \swarrow & & \searrow \epsilon_{\square} \\ \mathcal{H} & \xrightarrow{\epsilon} & k \end{array}$$

- *antipode: $S_{\square} : \mathcal{H}_{\square} \rightarrow \mathcal{H}_{\square}$, $S_{\square} := T^{-1} \circ S \circ T$,*

$$\begin{array}{ccc} \mathcal{H}_{\square} & \xrightarrow{S_{\square}} & \mathcal{H}_{\square} \\ T \downarrow & & \uparrow T^{-1} \\ \mathcal{H} & \xrightarrow{S} & \mathcal{H} \end{array}$$

- *character: $\xi_{\square} : \mathcal{H}_{\square} \rightarrow \mathcal{R}$, $\xi_{\square} := \xi \circ T$.*

$$\begin{array}{ccc} \mathcal{H}_{\square} & \xrightarrow{\xi_{\square}} & \mathcal{R} \\ T \searrow & & \nearrow \xi \\ & \mathcal{H} & \end{array}$$

The linear isomorphism T is obviously a Hopf algebra isomorphism.

Corollary 3.2. *In the notations of Proposition 3.1 we have:*

- a) *Let $\tilde{\mathcal{H}} \subset \mathcal{H}$ be a subalgebra of (\mathcal{H}, m, η) , then $T^{-1}(\tilde{\mathcal{H}}) \subset \mathcal{H}_{\square}$ is a subalgebra of $(\mathcal{H}_{\square}, m_{\square}, \eta_{\square})$.*

- b) Let $\tilde{\mathcal{H}} \subset \mathcal{H}$ be a subcoalgebra of $(\mathcal{H}, \Delta, \epsilon)$, then $T^{-1}(\tilde{\mathcal{H}}) \subset \mathcal{H}_\square$ is a subcoalgebra of $(\mathcal{H}_\square, \Delta_\square, \epsilon_\square)$.
c) Let $\tilde{\mathcal{H}} \subset \mathcal{H}$ be a (left) coideal of $(\mathcal{H}, \Delta, \epsilon)$, then $T^{-1}(\tilde{\mathcal{H}}) \subset \mathcal{H}_\square$ is a (left) coideal of $(\mathcal{H}_\square, \Delta_\square, \epsilon_\square)$.

4. APPLICATION OF T -HOPF ALGEBRA TO MZVS

Proposition 3.1 is applied in the context of classical multiple zeta values (MZVs). We briefly review the double-shuffle structure of MZVs and relate the dual product, which is derived from an involution defined on the shuffle Hopf algebra, to a derivation relation of MZVs.

4.1. Multiple zeta values. Thanks to the coexistence of sum and integral representations of MZVs, the \mathbb{Q} -algebra spanned by those values contains many linear relations, called *double shuffle relations*. The latter are most naturally described in terms of the algebraic framework of formal word algebras equipped with shuffle and quasi-shuffle products, which we review briefly.

We use the notations of Section 2. The *quasi-shuffle product* $m_*: \mathfrak{h}^1 \otimes \mathfrak{h}^1 \rightarrow \mathfrak{h}^1$, $u * v := m_*(u \otimes v)$ is iteratively defined as follows:

$$(QS1) \quad \mathbf{1} * w := w * \mathbf{1} := w$$

$$(QS2) \quad z_n u * z_m v := z_n(u * z_m v) + z_m(z_n u * v) + z_{n+m}(u * v)$$

for words $u, v, w \in \mathfrak{h}^1$ and $n, m \in \mathbb{N}$. Further the *shuffle product* $m_\sqcup: \mathfrak{h}^1 \otimes \mathfrak{h}^1 \rightarrow \mathfrak{h}^1$, $u \sqcup v := m_\sqcup(u \otimes v)$, is given by

$$(SH1) \quad \mathbf{1} \sqcup w := w \sqcup \mathbf{1} := w$$

$$(SH2) \quad au \sqcup bv := a(u \sqcup bv) + b(au \sqcup v)$$

for words $u, v, w \in \mathfrak{h}^1$ and letters $a, b \in \{x_0, x_1\}$. The difference between both products is seen best by comparing the quasi-shuffle product $z_2 * z_2 = 2z_2 z_2 + z_4$ with the shuffle product $x_0 x_1 \sqcup x_0 x_1 = 2x_0 x_1 x_0 x_1 + 4x_0 x_0 x_1 x_1$.

Defining the linear map $\zeta: \mathfrak{h}^0 \rightarrow \mathbb{R}$ for any convergent word $x_0^{k_1-1} x_1 \cdots x_0^{k_n-1} x_1 \in \mathfrak{h}^0$ by

$$\zeta(x_0^{k_1-1} x_1 \cdots x_0^{k_n-1} x_1) := \zeta(k_1, \dots, k_n),$$

we have the following well known result.

Theorem 4.1 ([24, 29, 11, 13]). *The map $\zeta: \mathfrak{h}^0 \rightarrow \mathbb{R}$ is an algebra morphism with respect to both algebras (\mathfrak{h}^0, m_*) and $(\mathfrak{h}^0, m_\sqcup)$. Moreover, the following so-called extended double shuffle relations hold*

$$(3) \quad \zeta(u * v - u \sqcup v) = 0 \quad \text{and} \quad \zeta(z_1 * w - x_1 \sqcup w) = 0,$$

for any words $u, v, w \in \mathfrak{h}^0$.

In fact, conjecturally all linear relations among MZVs are assumed to follow from the relations in (3). See [13].

4.2. Duality relations. With the notations of Section 2 at hand, we have the following well-known duality result:

Theorem 4.2 ([8]). *For any word $w \in \mathfrak{h}^0$ we have $\zeta(w) = \zeta(\tau(w))$.*

Finally, we consider the so-called *derivation relations*, which give rise to linear relations among MZVs. For $n \in \mathbb{N}$ we introduce the derivation $\partial_n: \mathfrak{h}^0 \rightarrow \mathfrak{h}^0$ by

$$\partial_n(x_0) := x_0(x_0 + x_1)^{n-1} x_1 \quad \text{and} \quad \partial_n(x_1) := -x_0(x_0 + x_1)^{n-1} x_1.$$

Then we have the next result.

Theorem 4.3 ([14]). *For any word $w \in \mathfrak{h}^0$ and any $n > 0$ we have $\zeta(\partial_n(w)) = 0$.*

Now we apply Proposition 3.1 to the quasi-shuffle Hopf algebra $(\mathfrak{h}^1, m_*, \Delta)$, where the coproduct $\Delta: \mathfrak{h}^1 \rightarrow \mathfrak{h}^1 \otimes \mathfrak{h}^1$ is defined by the usual deconcatenation of words

$$(4) \quad \Delta(z_{k_1} \cdots z_{k_n}) := z_{k_1} \cdots z_{k_n} \otimes \mathbf{1} + \mathbf{1} \otimes z_{k_1} \cdots z_{k_n} + \sum_{l=1}^{n-1} z_{k_1} \cdots z_{k_l} \otimes z_{k_{l+1}} \cdots z_{k_n}.$$

The product $m_\square: \mathfrak{h}^{(-1)} \otimes \mathfrak{h}^{(-1)} \rightarrow \mathfrak{h}^{(-1)}$ is defined in terms of $\tau: \mathfrak{h}^{(-1)} \rightarrow \mathfrak{h}^1$ and the quasi-shuffle product m_* on \mathfrak{h}^1

$$m_\square := \tau \circ m_* \circ (\tau \otimes \tau).$$

Since \mathfrak{h}^0 is a subalgebra of (\mathfrak{h}^1, m_*) , $(\mathfrak{h}^0, m_\square)$ is also a subalgebra of $(\mathfrak{h}^{(-1)}, m_\square)$ by Corollary 3.2. Note that Zudilin remarked in [30] that m_\square does not coincide with the usual shuffle product for MZVs defined algebraically through

(SH1) and (SH2). This can be seen by noticing that the weight is preserved under both duality as well as the quasi-shuffle product. Depth on the other hand is transformed by duality to the difference of weight and depth. As the quasi-shuffle product leads to a decrease in depth the product m_{\square} leads to an increase in depth, in contrast to the shuffle product, which preserves both weight and depth. For example

$$m_{\square}(x_0x_1 \otimes x_0x_1) = 2x_0x_1x_0x_1 + x_0x_1x_1x_1.$$

Now we show that the product m_{\square} is related to a derivation relation for MZVs.

Theorem 4.4. *Let $w \in \mathfrak{h}^0$. Then we have*

$$(5) \quad \partial_2(w) = w \square z_2 - w * z_2.$$

Remark 4.5. *One should compare Theorem 4.4 with an identity of Hoffman and Ohno proved in [11]. They showed that for any $w \in \mathfrak{h}^0$*

$$\partial_1(w) = w \sqcup z_1 - w * z_1.$$

Proof of Theorem 4.4. In the first step of the proof we show that (5) is true for $u = x_0^a x_1^b \in \mathfrak{h}^0$ with $a, b \in \mathbb{N}$.

- Let $a, b = 1$. Then $\partial_2(x_0x_1) = x_0x_1^3 - x_0^3x_1$ and

$$\begin{aligned} x_0x_1 \square x_0x_1 - z_2 * z_2 &= \tau(2x_0x_1x_0x_1 + x_0^3x_1) - (2x_0x_1x_0x_1 + x_0^3x_1) \\ &= x_0x_1^3 - x_0^3x_1. \end{aligned}$$

- Let $b = 1$. We obtain by induction hypothesis

$$\begin{aligned} \partial_2(x_0^a x_1) &= \partial_2(x_0)x_0^{a-1}x_1 + x_0\partial_2(x_0^{a-1}x_1) \\ &= x_0^2x_1x_0^{a-1}x_1 + x_0x_1^2x_0^{a-1}x_1 + x_0(x_0^{a-1}x_1 \square x_0x_1 - ((x_0^{a-1}x_1) * (x_0x_1))). \end{aligned}$$

Observing

$$\begin{aligned} x_0^a x_1 \square x_0x_1 &= \tau(z_2z_1^{a-1} * z_2) \\ &= \tau((z_2z_1^{a-2} * z_2)z_1) + \tau(z_2z_1^{a-1}z_2) + \tau(z_2z_1^{a-2}z_3) \\ &= x_0(x_0^{a-1}x_1 \square x_0x_1) + x_0x_1x_0^a x_1 + x_0x_1^2x_0^{a-1}x_1 \end{aligned}$$

and

$$\begin{aligned} x_0^a x_1 * x_0x_1 &= (z_{a+1} * z_2) = z_2z_{a+1} + z_{a+1}z_2 + z_{a+3} \\ &= x_0(z_az_2 + z_2z_a + z_{a+2}) - x_0(z_2z_a) + z_2z_{a+1} \\ &= x_0(z_a * z_2) - x_0^2x_1x_0^{a-1}x_1 + x_0x_1x_0^a x_1, \end{aligned}$$

concludes the proof.

- Let $a \in \mathbb{N}$ be fixed. Then by induction hypothesis

$$\begin{aligned} \partial_2(x_0^a x_1^b) &= \partial_2(x_0^a x_1^{b-1})x_1 + x_0^a x_1^{b-1} \partial_2(x_1) \\ &= (x_0^a x_1^{b-1} \square x_0x_1 - x_0^a x_1^{b-1} * x_0x_1)x_1 - x_0^a x_1^{b-1} x_0^2x_1 - x_0^a x_1^{b-1} x_0x_1^2. \end{aligned}$$

We obtain

$$\begin{aligned} x_0^a x_1^b \square x_0x_1 &= \tau(z_{b+1}z_1^{a-1} * z_2) \\ &= \tau(x_0(z_bz_1^{a-1} * z_2)) - \tau(x_0(z_2z_bz_1^{a-1})) + \tau(z_2z_{b+1}z_1^{a-1}) \\ &= (x_0^a x_1^{b-1} \square x_0x_1)x_1 - x_0^a x_1^{b-1} x_0x_1^2 + x_0^a x_1^b x_0x_1 \end{aligned}$$

and

$$\begin{aligned} x_0^a x_1^b * x_0x_1 &= (z_{a+1}z_1^{b-1} * z_2) \\ &= ((z_{a+1}z_1^{b-2} * z_2)z_1) + (z_{a+1}z_1^{b-1}z_2) + (z_{a+1}z_1^{b-2}z_3) \\ &= (x_0^a x_1^{b-1} * x_0x_1)x_1 + x_0^a x_1^b x_0x_1 + x_0^a x_1^{b-1} x_0^2x_1, \end{aligned}$$

which concludes the first part of the proof.

To finish the proof we define the map $\delta(u) := u \square x_0 x_1 - u * x_0 x_1$. For $w_1 := x_0^{a_1} x_1^{b_1} \cdots x_0^{a_n} x_1^{b_n}$ and $w_2 := x_0^{c_1} x_1^{d_1} \cdots x_0^{c_m} x_1^{d_m}$ with $m, n \in \mathbb{N}$ we will prove that

$$(6) \quad \delta(w_1 w_2) = \delta(w_1) w_2 + w_1 \delta(w_2).$$

Indeed, we have

$$\begin{aligned} w_1 w_2 \square x_0 x_1 &= \tau(z_{d_m+1} z_1^{c_m-1} \cdots z_{d_1+1} z_1^{c_1-1} z_{b_n+1} z_1^{a_n-1} \cdots z_{b_1+1} z_1^{a_1-1} * z_2) \\ &= \tau(z_{b_n+1} z_1^{a_n-1} \cdots z_{b_1+1} z_1^{a_1-1}) \tau(z_{d_m+1} z_1^{c_m-1} \cdots z_{d_1+1} z_1^{c_1-1} * z_2) \\ &\quad + \tau(z_{b_n+1} z_1^{a_n-1} \cdots z_{b_1+1} z_1^{a_1-1} * z_2) \tau(z_{d_m+1} z_1^{c_m-1} \cdots z_{d_1+1} z_1^{c_1-1}) \\ &\quad - \tau(z_{d_m+1} z_1^{c_m-1} \cdots z_{d_1+1} z_1^{c_1-1} z_2 z_{b_n+1} z_1^{a_n-1} \cdots z_{b_1+1} z_1^{a_1-1}) \\ &= w_1(w_2 \square x_0 x_1) + (w_1 \square x_0 x_1) w_2 - w_1 x_0 x_1 w_2 \end{aligned}$$

as well as

$$\begin{aligned} w_1 w_2 * x_0 x_1 &= z_{a_1+1} z_1^{b_1-1} \cdots z_{a_n+1} z_1^{b_n-1} z_{c_1+1} z_1^{d_1-1} \cdots z_{c_m+1} z_1^{d_m-1} * z_2 \\ &= (z_{a_1+1} z_1^{b_1-1} \cdots z_{a_n+1} z_1^{b_n-1}) (z_{c_1+1} z_1^{d_1-1} \cdots z_{c_m+1} z_1^{d_m-1} * z_2) \\ &\quad + (z_{a_1+1} z_1^{b_1-1} \cdots z_{a_n+1} z_1^{b_n-1} * z_2) (z_{c_1+1} z_1^{d_1-1} \cdots z_{c_m+1} z_1^{d_m-1}) \\ &\quad - (z_{a_1+1} z_1^{b_1-1} \cdots z_{a_n+1} z_1^{b_n-1} z_2 z_{c_1+1} z_1^{d_1-1} \cdots z_{c_m+1} z_1^{d_m-1}) \\ &= w_1(w_2 * x_0 x_1) + (w_1 * x_0 x_1) w_2 - w_1 x_0 x_1 w_2. \end{aligned}$$

This implies (6). Combining the first and the second step proves the claim of the theorem. \square

5. APPLICATION OF T -HOPF ALGEBRA TO THE SHUFFLE PRODUCT OF q -MZVS

A systematic study of q -analogues of MZVs (q -MZVs) was initiated by Bradley, Zhao and Zudilin [2, 27, 29], with a forerunner by Schlesinger's 2001 preprint [20]. Since then several distinct q -models of MZVs have been considered in the literature. See Zhao's recent work [28] for details. The quasi-shuffle products of these q -MZVs are deduced from the defining series, and in several cases an accompanying Hopf algebra structure in the sense of Hoffman [10] can be defined. Quite recently q -shuffle products for the most prevailing models of q -MZVs were studied in [4, 5, 21, 22]. See also [28]. They are based on a description of q -MZVs in terms of iterated Rota–Baxter operators. The latter enter the picture through substituting Jackson's q -integral for the Riemann integral in the integral representation of ordinary MZVs.

In this section we compare the Rota–Baxter operator approach and the duality approach for the q -analogues of Schlesinger–Zudilin (SZ-) model and Ohno–Okuda–Zudilin (OOZ-) model. In the forthcoming we always assume that $q \in \mathbb{C}$ with $|q| < 1$. The notation $q \uparrow 1$ stands for $q \rightarrow 1$ inside some angular sector $-\pi/2 + \varepsilon < \text{Arg}(1 - q) < \pi/2 - \varepsilon$ centered at 1, with a fixed $\varepsilon > 0$.

5.1. Schlesinger–Zudilin q -model. In [20] and [29] the authors defined the following q -analogues of MZVs given by

$$(7) \quad \zeta^{\text{SZ}}(k_1, \dots, k_n) := \sum_{m_1 \geq \dots \geq m_n > 0} \frac{q^{m_1 k_1 + \dots + m_n k_n}}{(1 - q^{m_1})^{k_1} \cdots (1 - q^{m_n})^{k_n}},$$

for $k_1, \dots, k_n \in \mathbb{N}_0$ with $k_1 \geq 1$. It is easily seen that for $k_1 \geq 2$ and $k_2, \dots, k_n \geq 1$ we obtain

$$\lim_{q \uparrow 1} (1 - q)^{k_1 + \dots + k_n} \zeta^{\text{SZ}}(k_1, \dots, k_n) = \zeta(k_1, \dots, k_n).$$

Furthermore we introduce a star version of model (7) which is given for any $k_1, \dots, k_n \in \mathbb{N}_0$ with $k_1 \geq 1$ by

$$(8) \quad \zeta^{\text{SZ},*}(k_1, \dots, k_n) := \sum_{m_1 \geq \dots \geq m_n > 0} \frac{q^{m_1 k_1 + \dots + m_n k_n}}{(1 - q^{m_1})^{k_1} \cdots (1 - q^{m_n})^{k_n}}.$$

This model plays a distinguished role with respect to the OOOZ-model, which will be discussed in the next paragraph. Again we observe for $k_1 \geq 2$ and $k_2, \dots, k_n \geq 1$ that

$$\lim_{q \uparrow 1} (1 - q)^{k_1 + \dots + k_n} \zeta^{\text{SZ},*}(k_1, \dots, k_n) = \zeta^*(k_1, \dots, k_n) := \sum_{m_1 \geq \dots \geq m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

where $\zeta^*(k_1, \dots, k_n)$ are called *multiple zeta star values* (MZSVs).

Remark 5.1. Note that (7) constitutes what is usually called a modified (Schlesinger–Zudilin) q -MZV, and the proper q -MZV is given by $(1-q)^{k_1+\dots+k_n} \zeta^{\text{SZ}}(k_1, \dots, k_n)$. The same applies to the q -MZSV defined in (8). In the following we will work exclusively with (7) and (8).

Using the notations of Section 2 again, we define the λ -weighted quasi-shuffle type product $m_*^{(\lambda)}: \mathfrak{H}^1 \otimes \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$, $u *_{\lambda} v := m_*^{(\lambda)}(u \otimes v)$, iteratively by

$$(QS1) \quad \mathbf{1} *_{\lambda} w := w *_{\lambda} \mathbf{1} := w,$$

$$(QS2) \quad z_n u *_{\lambda} z_m v := z_n(u *_{\lambda} z_m v) + z_m(z_n u *_{\lambda} v) + \lambda z_{n+m}(u *_{\lambda} v)$$

for any words $w, u, v \in \mathfrak{H}^1$ and $n, m \in \mathbb{N}_0$. The parameter λ will stand for 1 or -1 . The case $\lambda = 1$ corresponds to the natural product satisfied by the q -MZVs defined in (7). It coincides with the quasi-shuffle product of classical MZVs. Note that $(\mathfrak{H}^1, m_*^{(1)}, \Delta)$, where Δ is given by deconcatenation (4), defines a quasi-shuffle Hopf algebra. The case $\lambda = -1$ corresponds to the product satisfied by the q -MZSVs defined in (8). Now we introduce the λ -weighted shuffle product $m_{\sqcup}^{(\lambda)}: \mathfrak{H}^1 \otimes \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$, $u \sqcup_{\lambda} v := m_{\sqcup}^{(\lambda)}(u \otimes v)$, by

$$(SH1) \quad \mathbf{1} \sqcup_{\lambda} w := w \sqcup_{\lambda} \mathbf{1} := w,$$

$$(SH2) \quad yu \sqcup_{\lambda} v := u \sqcup_{\lambda} yv := y(u \sqcup_{\lambda} v),$$

$$(SH3) \quad pu \sqcup_{\lambda} pv := p(u \sqcup_{\lambda} pv) + p(pu \sqcup_{\lambda} v) + \lambda p(u \sqcup_{\lambda} v).$$

Defining the map $\zeta^{\text{SZ}}: \mathfrak{H}^0 \rightarrow \mathbb{Q}[[q]]$ by

$$\zeta^{\text{SZ}}(z_{k_1} \cdots z_{k_n}) := \zeta^{\text{SZ}}(k_1, \dots, k_n)$$

we can state the following result of [21]:

Theorem 5.2. The map $\zeta^{\text{SZ}}: \mathfrak{H}^0 \rightarrow \mathbb{Q}[[q]]$ is an algebra morphism on both algebras $(\mathfrak{H}^0, m_*^{(1)})$ and $(\mathfrak{H}^0, m_{\sqcup}^{(1)})$. Especially, for any words $u, v \in \mathfrak{H}^0$ we have the q -analogue of the double shuffle relation

$$\zeta^{\text{SZ}}(u \sqcup_1 v - u *_1 v) = 0.$$

First we introduce the linear map $\tilde{\tau}: \mathbb{Q}\langle p, y \rangle \rightarrow \mathbb{Q}\langle p, y \rangle$

$$(9) \quad \tilde{\tau}(p) := y \quad \tilde{\tau}(y) := p,$$

which is extended to words as an antiautomorphism. Again, this induces the linear isomorphism $\tilde{\tau}: \mathfrak{H}^{(-1)} \rightarrow \mathfrak{H}^1$. By further restriction we obtain the automorphism $\tilde{\tau}: \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$, which yields the following duality result due to Zhao.

Theorem 5.3. [28, Theo. 8.3] For any word $w \in \mathfrak{H}^0$ we have $\zeta^{\text{SZ}}(w) = \zeta^{\text{SZ}}(\tilde{\tau}(w))$.

The first nontrivial relation arising from Theorem 5.3 is:

$$\zeta^{\text{SZ}}(2) = \zeta^{\text{SZ}}(ppy) = \zeta^{\text{SZ}}(pyy) = \zeta^{\text{SZ}}(1, 0) = \sum_{k, l > 0} (k-1)q^{kl}.$$

Proof. We include brief demonstration of Theorem 5.3. As in [21] we introduce the Rota–Baxter operator

$$\overline{P}_q[f](t) := \sum_{m \geq 1} f(q^m t)$$

of weight one. Further let $y(t) := \frac{t}{1-t} \in t\mathbb{Q}[[t]]$. Using the identity [21, Prop. 2.6 (i)]

$$\zeta^{\text{SZ}}(p^{k_1} y \cdots p^{k_n} y) = \overline{P}_q^{k_1}[y \overline{P}_q^{k_2}[y \cdots \overline{P}_q^{k_n}[y] \cdots]](t) \Big|_{t=1}$$

we calculate

$$\begin{aligned} \zeta^{\text{SZ}}(\tau_q(p^{k_1} y \cdots p^{k_n} y)) &= \zeta^{\text{SZ}}(py^{k_n} \cdots py^{k_1}) \\ &= \overline{P}_q[y^{k_n} \overline{P}_q[y^{k_{n-1}} \cdots \overline{P}_q[y^{k_1}]]](t) \Big|_{t=1} \\ &= \sum_{m_1, \dots, m_n > 0} \frac{q^{(m_1+\dots+m_n)k_1} t^{k_1}}{(1-q^{m_1+\dots+m_n} t)^{k_1}} \frac{q^{(m_2+\dots+m_n)k_2} t^{k_2}}{(1-q^{m_2+\dots+m_n} t)^{k_2}} \cdots \frac{q^{m_n k_n} t^{k_n}}{(1-q^{m_n} t)^{k_n}} \Big|_{t=1} \\ &= \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{q^{m_1 k_1 + \dots + m_n k_n}}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_n})^{k_n}} \\ &= \zeta^{\text{SZ}}(p^{k_1} y \cdots p^{k_n} y). \end{aligned}$$

□

Again, we are in the position to apply Proposition 3.1 in the context of the quasi-shuffle Hopf algebra $(\mathfrak{H}^1, m_*^{(1)}, \Delta)$ and the character ζ^{SZ} . Since \mathfrak{H}^0 is a subalgebra of \mathfrak{H}^1 it is also a subalgebra of \mathfrak{H}_\square^1 , and we obtain the algebra $(\mathfrak{H}^0, m_\square^{(1)})$. Here we find the same situation as in [30], i.e., the product $m_\square^{(1)} = \tilde{\tau} \circ m_*^{(1)} \circ (\tilde{\tau} \otimes \tilde{\tau})$, $a \sqcup b := m_\square^{(1)}(a \otimes b)$ equals the shuffle product $m_\square^{(1)}$.

Theorem 5.4. *The product $m_\square^{(1)}: \mathfrak{H}^0 \otimes \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$ induced by Proposition 3.1 coincides with $m_\square^{(1)}: \mathfrak{H}^0 \otimes \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$. In particular, we have the double shuffle relations*

$$\zeta^{\text{SZ}}(u \sqcup v - u * v) = 0$$

for any words $u, v \in \mathfrak{H}^0$.

Proof. Let $X := \{p, y\}$. For $u \in \mathfrak{H}^0$ we observe that $\mathbf{1} \sqcup u = u \sqcup \mathbf{1} = \tilde{\tau}(\mathbf{1} * \tilde{\tau}(u)) = u$. Let $u', v' \in pX^*y$ then there exist $a, b, c, d \geq 1$ and $u, v \in pX^*y \cup \{\mathbf{1}\}$ such that $u' = p^a y^b u$ and $v' = p^c y^d v$. We observe that the product

$$p^a y^b u \sqcup p^c y^d v = \tilde{\tau}(\tilde{\tau}(p^a y^b u) * \tilde{\tau}(p^c y^d v)) = \tilde{\tau}(\tilde{\tau}(u) z_b z_0^{a-1} * \tilde{\tau}(v) z_d z_0^{c-1}).$$

Now we distinguish three cases:

- Case 1: $a = c = 1$

$$\begin{aligned} p y^b u \sqcup p y^d v &= \tilde{\tau}(\tilde{\tau}(u) z_b * \tilde{\tau}(v) z_d) \\ &= \tilde{\tau}\{(\tilde{\tau}(u) * \tilde{\tau}(v) z_d) z_b + (\tilde{\tau}(u) z_b * \tilde{\tau}(v)) z_d + (\tilde{\tau}(u) * \tilde{\tau}(v)) z_{b+d}\} \\ &= p y^b(u \sqcup p y^d v) + p y^d(p y^b u \sqcup v) + p y^{b+d}(u \sqcup v). \end{aligned}$$

- Case 2: $a = 1, c \geq 2$

$$\begin{aligned} p y^b u \sqcup p^c y^d v &= \tilde{\tau}(\tilde{\tau}(u) z_b * \tilde{\tau}(v) z_d z_0^{c-1}) \\ &= \tilde{\tau}\{(\tilde{\tau}(u) * \tilde{\tau}(v) z_d z_0^{c-1}) z_b + (\tilde{\tau}(u) z_b * \tilde{\tau}(v) z_d z_0^{c-2}) z_0 + (\tilde{\tau}(u) * \tilde{\tau}(v) z_d z_0^{c-2}) z_b\} \\ &= p y^b(u \sqcup p^c y^d v) + p(p y^b u \sqcup p^{c-1} y^d v) + p y^b(u \sqcup p^{c-1} y^d v). \end{aligned}$$

The case $a \geq 2, c = 1$ is analogues.

- Case 3: $a, c \geq 2$

$$\begin{aligned} p^a y^b u \sqcup p^c y^d v &= \tilde{\tau}(\tilde{\tau}(u) z_b z_0^{a-1} * \tilde{\tau}(v) z_d z_0^{c-1}) \\ &= \tilde{\tau}\{(\tilde{\tau}(u) z_b z_0^{a-2} * \tilde{\tau}(v) z_d z_0^{c-1}) z_0 + (\tilde{\tau}(u) z_b z_0^{a-1} * \tilde{\tau}(v) z_d z_0^{c-2}) z_0 + (\tilde{\tau}(u) z_b z_0^{a-2} * \tilde{\tau}(v) z_d z_0^{c-2}) z_b\} \\ &= p(p^{a-1} y^b u \sqcup p^c y^d v) + p(p^a y^b u \sqcup p^{c-1} y^d v) + p(p^{a-1} y^b u \sqcup p^{c-1} y^d v). \end{aligned}$$

It is easy to see that the three cases reduce to (SH1), (SH2) and (SH3), respectively. □

5.2. Ohno–Okuda–Zudilin q -model. Now we consider another q -model introduced by Ohno, Okuda and Zudilin (OOZ) in [19]. Let $k_1, \dots, k_n \in \mathbb{Z}$. Then the (modified) OOZ q -MZVs are defined by

$$(10) \quad \zeta^{\text{OOZ}}(k_1, \dots, k_n) := \sum_{m_1 > \dots > m_n > 0} \frac{q^{m_1}}{(1 - q^{m_1})^{k_1} \dots (1 - q^{m_n})^{k_n}}.$$

Again, this is a proper q -analogue of MZVs thanks to the fact that the limit

$$\lim_{q \uparrow 1} (1 - q)^{k_1 + \dots + k_n} \zeta^{\text{OOZ}}(k_1, \dots, k_n) = \zeta(k_1, \dots, k_n),$$

if $k_1 \geq 2, k_2, \dots, k_n \geq 1$.

If we restrict ourselves to non-negative integers then we can introduce an algebraic setting analogues to the one for the Schlesinger–Zudilin model presented in the previous section. We define

$$\zeta^{\text{OOZ}}: \mathfrak{H}^0 \rightarrow \mathbb{Q}[[q]], \quad \zeta^{\text{OOZ}}(z_{k_1} \dots z_{k_n}) := \zeta^{\text{OOZ}}(k_1, \dots, k_n)$$

and

$$\zeta^{\text{SZ},*}: \mathfrak{H}^0 \rightarrow \mathbb{Q}[[q]], \quad \zeta^{\text{SZ},*}(z_{k_1} \dots z_{k_n}) := \zeta^{\text{SZ},*}(k_1, \dots, k_n).$$

Using the linear map $\tilde{\tau}$ defined in (9) we can state the following duality theorem.

Theorem 5.5. *For any word $w \in \mathfrak{H}^0$ we find that $\zeta^{\text{OOZ}}(w) = \zeta^{\text{SZ},*}(\tilde{\tau}(w))$.*

Proof. We use the Rota–Baxter operator $P_q = \text{Id} + \overline{P}_q$, which explicitly writes

$$(11) \quad P_q[f](t) := \sum_{m \geq 0} f(q^m t)$$

of weight -1 , together with $y(t) := \frac{t}{1-t} \in t\mathbb{Q}[[t]]$. Following [5, Eq. (4.1)] we can write

$$\zeta^{\text{OOZ}}(p^{k_1} y \cdots p^{k_n} y) = P_q^{k_1} [y P_q^{k_2} [y \cdots P_q^{k_n} [y] \cdots]](t) \Big|_{t=q},$$

and calculate

$$\begin{aligned} \zeta^{\text{OOZ}}(\tilde{\tau}(p^{k_1} y \cdots p^{k_n} y)) &= \zeta^{\text{OOZ}}(p y^{k_n} \cdots p y^{k_1}) \\ &= P_q[y^{k_n} P_q[y^{k_{n-1}} \cdots P_q[y^{k_1}]](t) \Big|_{t=q} \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{q^{(m_1 + \cdots + m_n)k_1} t^{k_1}}{(1 - q^{m_1 + \cdots + m_n} t)^{k_1}} \frac{q^{(m_2 + \cdots + m_n)k_2} t^{k_2}}{(1 - q^{m_2 + \cdots + m_n} t)^{k_2}} \cdots \frac{q^{m_n k_n} t^{k_n}}{(1 - q^{m_n} t)^{k_n}} \Big|_{t=q} \\ &= \sum_{m_1 \geq m_2 \geq \cdots \geq m_n > 0} \frac{q^{m_1 k_1 + \cdots + m_n k_n}}{(1 - q^{m_1})^{k_1} \cdots (1 - q^{m_n})^{k_n}} \\ &= \zeta^{SZ, \star}(p^{k_1} y \cdots p^{k_n} y). \end{aligned}$$

Since $\tilde{\tau}$ is an antiautomorphism for words from $\mathbb{Q}\langle p, y \rangle$, the proof is complete. \square

In general $\tilde{\tau}$ does not preserve the weight of a given word. Therefore we can not deduce the classical duality relation of MZVs stated in Theorem 4.2 by taking the limit $q \uparrow 1$ in the previous theorem. Subsequently, we prove that in leading q -order a q -version of Theorem 4.2 for the OOOZ-model is valid (Corollary 5.11). Recall that for $k_1 \geq 2$, $k_2, \dots, k_n \geq 1$ the (modified) Bradley–Zhao (BZ) q -MZVs are defined by the iterated sum

$$\zeta^{\text{BZ}}(k_1, \dots, k_n) := \sum_{m_1 > \cdots > m_n > 0} \frac{q^{(k_1-1)m_1 + \cdots + (k_n-1)m_n}}{(1 - q^{m_1})^{k_1} \cdots (1 - q^{m_n})^{k_n}}.$$

For this model we have the following well-known result:

Theorem 5.6. [2] *For any word $w \in \mathfrak{h}^0$ we have $\zeta^{\text{BZ}}(w) = \zeta^{\text{BZ}}(\tau(w))$.*

The notation $\zeta^{\text{BZ}}(z_{k_1} \cdots z_{k_n}) = \zeta^{\text{BZ}}(x_0 x_1^{k_1-1} \cdots x_0 x_1^{k_n-1}) := \zeta^{\text{BZ}}(k_1, \dots, k_n)$ is used. It will be useful to reformulate both Theorems 5.3 and 5.6 in the OOOZ model. Let us introduce the linear maps $U : \mathfrak{h}^0 \rightarrow \mathfrak{h}^0$ and $V : \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$ defined by

$$(12) \quad U(z_{k_1} \cdots z_{k_n}) := \sum_{\substack{2 \leq r_1 \leq k_1 \\ 1 \leq r_j \leq k_j, j \geq 2}} \binom{k_1-2}{r_1-2} \binom{k_2-1}{r_2-1} \cdots \binom{k_n-1}{r_n-1} z_{r_1} \cdots z_{r_n}$$

and

$$(13) \quad V(z_{k_1} \cdots z_{k_n}) := \sum_{\substack{1 \leq r_1 \leq k_1 \\ 0 \leq r_j \leq k_j, j \geq 2}} \binom{k_1-1}{r_1-1} \binom{k_2}{r_2} \cdots \binom{k_n}{r_n} z_{r_1} \cdots z_{r_n}.$$

Proposition 5.7. *We have the following:*

(1) *The maps U and V are linear isomorphisms, with inverses given by*

$$(14) \quad U^{-1}(z_{k_1} \cdots z_{k_n}) := \sum_{\substack{2 \leq r_1 \leq k_1 \\ 1 \leq r_j \leq k_j, j \geq 2}} (-1)^{\sum_j k_j - r_j} \binom{k_1-2}{r_1-2} \binom{k_2-1}{r_2-1} \cdots \binom{k_n-1}{r_n-1} z_{r_1} \cdots z_{r_n}$$

and

$$(15) \quad V^{-1}(z_{k_1} \cdots z_{k_n}) := \sum_{\substack{1 \leq r_1 \leq k_1 \\ 0 \leq r_j \leq k_j, j \geq 2}} (-1)^{\sum_j k_j - r_j} \binom{k_1-1}{r_1-1} \binom{k_2}{r_2} \cdots \binom{k_n}{r_n} z_{r_1} \cdots z_{r_n}.$$

(2) *For convergent words, the Bradley–Zhao and Ohno–Okuda–Zudilin models are related by*

$$(16) \quad \zeta^{\text{OOZ}} = \zeta^{\text{BZ}} \circ U.$$

(3) For words in \mathfrak{H}^0 , the Schlesinger–Zudilin and Ohno–Okuda–Zudilin models are related by

$$(17) \quad \zeta^{\text{OOZ}} = \zeta^{\text{SZ}} \circ V.$$

Proof. Let U' , respectively V' , be the linear endomorphism of \mathfrak{h}^0 , respectively \mathfrak{H}^0 , given by the right-hand side of (14), respectively (15). We compute

$$\begin{aligned} (U' \circ U)(z_{k_1} \cdots z_{k_n}) &= \sum_{\substack{2 \leq r_1 \leq k_1 \\ 1 \leq r_j \leq k_j, j \geq 2}} \binom{k_1-2}{r_1-2} \binom{k_2-1}{r_2-1} \cdots \binom{k_n-1}{r_n-1} U'(z_{r_1} \cdots z_{r_n}) \\ &= \sum_{\substack{2 \leq s_1 \leq r_1 \leq k_1 \\ 1 \leq s_j \leq r_j \leq k_j, j \geq 2}} (-1)^{\sum_j r_j - s_j} \binom{k_1-2}{r_1-2} \binom{k_2-1}{r_2-1} \cdots \binom{k_n-1}{r_n-1} \binom{r_1-2}{s_1-2} \binom{r_2-1}{s_2-1} \cdots \binom{r_n-1}{s_n-1} z_{s_1} \cdots z_{s_n} \\ &= \sum_{\substack{2 \leq s_1 \leq k_1 \\ 1 \leq s_j \leq k_j, j \geq 2}} D_{k_1-2}^{s_1-2} D_{k_2-1}^{s_2-1} \cdots D_{k_n-1}^{s_n-1} z_{s_1} \cdots z_{s_n}, \end{aligned}$$

with $D_k^a := \sum_{a+b+c=k} (-1)^b \frac{k!}{a!b!c!}$. It is easily seen by expanding $x^k = (x-y+y)^k$ that D_k^a reduces to the Kronecker symbol δ_k^a . Hence $(U' \circ U)(z_{k_1} \cdots z_{k_n}) = z_{k_1} \cdots z_{k_n}$. The verifications of $U \circ U' = \text{Id}_{\mathfrak{h}^0}$, $V' \circ V = \text{Id}_{\mathfrak{H}^0}$ and $V \circ V' = \text{Id}_{\mathfrak{H}^0}$ are entirely similar. The second assertion easily follows from the two following identities:

$$\frac{q^m}{(1-q^m)^k} = \sum_{l=2}^k \binom{k-2}{l-2} \frac{q^{(l-1)m}}{(1-q^m)^l} \quad \text{and} \quad \frac{1}{(1-q^m)^k} = \sum_{l=1}^k \binom{k-1}{l-1} \frac{q^{(l-1)m}}{(1-q^m)^l}.$$

The third assertion follows from:

$$\frac{q^m}{(1-q^m)^k} = \sum_{l=1}^k \binom{k-1}{l-1} \frac{q^{lm}}{(1-q^m)^l} \quad \text{and} \quad \frac{1}{(1-q^m)^k} = \sum_{l=0}^k \binom{k}{l} \frac{q^{lm}}{(1-q^m)^l}.$$

□

Remark 5.8. We obviously have

$$\zeta^{\text{BZ}} = \zeta^{\text{OOZ}} \circ U^{-1}, \quad \zeta^{\text{SZ}} = \zeta^{\text{OOZ}} \circ V^{-1}.$$

This could have been directly checked through the following identities:

$$\begin{aligned} \frac{q^{(k-1)m}}{(1-q^m)^k} &= \sum_{l=2}^k (-1)^{k-l} \binom{k-2}{l-2} \frac{q^m}{(1-q^m)^l}, & \frac{q^{(k-1)m}}{(1-q^m)^k} &= \sum_{l=1}^k (-1)^{k-l} \binom{k-1}{l-1} \frac{1}{(1-q^m)^l}, \\ \frac{q^{km}}{(1-q^m)^k} &= \sum_{l=1}^k (-1)^{k-l} \binom{k-1}{l-1} \frac{q^m}{(1-q^m)^l}, & \frac{q^{km}}{(1-q^m)^k} &= \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} \frac{1}{(1-q^m)^l}. \end{aligned}$$

Hence we obtain two families of duality relations in the OOZ model:

Theorem 5.9. Let $w \in \mathfrak{H}^0$. Then

$$\zeta^{\text{OOZ}}(w) = \zeta^{\text{OOZ}}(V^{-1} \circ \tilde{\tau} \circ V(w)).$$

If moreover $w \in \mathfrak{h}^0$ we also have

$$\zeta^{\text{OOZ}}(w) = \zeta^{\text{OOZ}}(U^{-1} \circ \tau \circ U(w)).$$

For example, from $V^{-1} \circ \tilde{\tau} \circ V(z_3) = z_1 z_0^2 + 2z_1 z_0 + z_1$ and $U^{-1} \circ \tau \circ U(z_3) = z_2 z_1 + z_2$ we get:

$$\begin{aligned} \zeta^{\text{OOZ}}(3) &= \zeta^{\text{OOZ}}(1) + 2\zeta^{\text{OOZ}}(1,0) + \zeta^{\text{OOZ}}(1,0,0) \\ &= \zeta^{\text{OOZ}}(2,1) + \zeta^{\text{OOZ}}(2) \\ &= \sum_{k,l \geq 0} \frac{k(k+1)}{2} q^{kl}. \end{aligned}$$

The second equality can also be obtained as a double shuffle relation [5]. Theorems 5.3 and 5.6 can be reformulated as follows:

Corollary 5.10. *For $a_1, b_1, \dots, a_n, b_n \in \mathbb{N}$ we have*

$$\begin{aligned}\zeta^{\text{SZ}}(a_1, \{0\}^{b_1-1}, \dots, a_n, \{0\}^{b_n-1}) &= \zeta^{\text{SZ}}(b_n, \{0\}^{a_n-1}, \dots, b_1, \{0\}^{a_1-1}), \\ \zeta^{\text{BZ}}(a_1 + 1, \{1\}^{b_1-1}, \dots, a_n + 1, \{1\}^{b_n-1}) &= \zeta^{\text{BZ}}(b_n + 1, \{1\}^{a_n-1}, \dots, b_1 + 1, \{1\}^{a_1-1}).\end{aligned}$$

Using this reformulation and Theorem 5.9 we immediately get:

Corollary 5.11. *We have*

$$\begin{aligned}\zeta^{\text{OOZ}}(a_1 + 1, \{1\}^{b_1-1}, \dots, a_n + 1, \{1\}^{b_n-1}) &= \zeta^{\text{OOZ}}(b_n + 1, \{1\}^{a_n-1}, \dots, b_1 + 1, \{1\}^{a_1-1}) + A(q), \\ \zeta^{\text{OOZ}}(a_1, \{0\}^{b_1-1}, \dots, a_n, \{0\}^{b_n-1}) &= \zeta^{\text{OOZ}}(b_n, \{0\}^{a_n-1}, \dots, b_1, \{0\}^{a_1-1}) + B(q),\end{aligned}$$

where $A(q) \in \langle \zeta^{\text{OOZ}}(k_1, \dots, k_n) : k_1 \geq 2, k_2, \dots, k_n \geq 1 \rangle_{\mathbb{Q}}$ and $B(q) \in \langle \zeta^{\text{OOZ}}(k_1, \dots, k_n) : k_1 \geq 1, k_2, \dots, k_n \geq 0 \rangle_{\mathbb{Q}}$. Moreover we have

$$\lim_{q \uparrow 1} (1 - q)^{a_1 + b_1 + \dots + a_n + b_n} A(q) = 0.$$

Proof. This is immediate. The last assertion on the term $A(q)$ is derived from the triangular structure of U , i.e., $U(w) = w + \text{terms of strictly lower weight}$. The term $B(q)$ is not so nice, due to the fact that the duality $\tilde{\tau}$ does not preserve the weight. \square

Let us go back to the duality construction of Theorem 5.5. For the OZ-model we apply Proposition 3.1 in the context of the Hopf algebra $(\mathfrak{H}^1, m_*^{(-1)}, \Delta)$ with character $\zeta^{\text{SZ},*}$, where Δ is defined as in (4). We obtain the dual Hopf algebra $(\mathfrak{H}^{(-1)}, m_{\square}^{(-1)}, \Delta_{\square})$. Again, \mathfrak{H}^0 is a subalgebra of $(\mathfrak{H}^{(-1)}, m_{\square}^{(-1)})$ and we have the following theorem.

Theorem 5.12. *The product $m_{\square}^{(-1)} : \mathfrak{H}^0 \otimes \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$ induced by Proposition 3.1 coincides with the shuffle product $m_{\square}^{(-1)} : \mathfrak{H}^0 \otimes \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$.*

Proof. The proof is very similar to that of Theorem 5.4. We only have to consider the weight -1 quasi-shuffle and shuffle products instead of the weight 1 products. \square

Remark 5.13. *Note that the q -MZVs in the OZ-model (10) are well defined for any integer arguments. However, we are not able to extend the duality argument of Theorem 5.12 to arbitrary integers, and have to restrict to non-negative arguments. The shuffle product approach provided in [5] is valid for any integer. The reason for this is that the Rota–Baxter operator P_q defined in (11) is invertible. This fact was used in [6] to provide a renormalization of MZVs with respect to the shuffle product, which is compatible with meromorphic continuation of MZVs.*

5.3. Comparison of different q -models. In this paragraph we discuss the connection of the shuffle products for the SZ- and OZ-models. The duality construction unveils that the link corresponds to the interplay between multiple zeta and multiple zeta star values.

Following [12] we define the map $S : \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$ recursively by $S(\mathbf{1}) = \mathbf{1}$ and

$$S(z_k w) = z_k S(w) + z_k \circ S(w)$$

for any $k \in \mathbb{N}$ and $w \in \mathfrak{H}^1$. The composition $\circ : \mathfrak{H}^1 \otimes \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$ is defined by $z_k \circ \mathbf{1} = 0$ and

$$z_{k_1} \circ (z_{k_2} w) = z_{k_1 + k_2} w$$

for any $w \in \mathfrak{H}^1$ and $k_1, k_2 \in \mathbb{N}_0$.

Then we have the following result:

Theorem 5.14. [12] *We have:*

- *The map S is an isomorphism with inverse S^{-1} given by $S^{-1}(\mathbf{1}) = \mathbf{1}$*

$$S^{-1}(z_k w) = z_k S^{-1}(w) - z_k \circ S^{-1}(w)$$

for any $k \in \mathbb{N}_0$ and $w \in \mathfrak{H}^1$.

- *The map $S : (\mathfrak{H}^1, m_*^{(-1)}) \rightarrow (\mathfrak{H}^1, m_*^{(1)})$ is an algebra isomorphism, i.e.,*

$$S(u *_{-1} v) = S(u) *_1 S(v)$$

for any $u, v \in \mathfrak{H}^1$.

Therefore, putting these results together, we obtain:

$$\begin{array}{ccc}
\mathfrak{H}^0 \otimes \mathfrak{H}^0 & \xrightarrow{\sqcup_{-1}} & \mathfrak{H}^0 \\
\tilde{\tau} \otimes \tilde{\tau} \downarrow & & \uparrow \tilde{\tau} \\
\mathfrak{H}^0 \otimes \mathfrak{H}^0 & \xrightarrow{*_{-1}} & \mathfrak{H}^0 \\
S \otimes S \downarrow & & \downarrow S \\
\mathfrak{H}^0 \otimes \mathfrak{H}^0 & \xrightarrow{*_1} & \mathfrak{H}^0 \\
\tilde{\tau} \otimes \tilde{\tau} \uparrow & & \downarrow \tilde{\tau} \\
\mathfrak{H}^0 \otimes \mathfrak{H}^0 & \xrightarrow{\sqcup_1} & \mathfrak{H}^0
\end{array}$$

Corollary 5.15. *The following diagram commutes:*

Proof. From top to bottom the commutativity of the diagram is ensured by Theorems 5.12, 5.14 and 5.4. \square

6. FURTHER APPLICATIONS OF THE T -HOPF ALGEBRA TO q -MZVS

6.1. A shuffle-like bialgebra structure for the OOOZ-model. In [4, 5] the authors provided a description of the shuffle product for the OOOZ-model. The main ingredient is a Rota–Baxter operator (RBO) approach, which substitutes for the Kontsevich integral formula in the classical MZVs case. Since the corresponding RBO is invertible, the shuffle product can be extended to arbitrary integer arguments.

First we quickly review the algebra framework. Let $X := \{p, d, y\}$, and let W denote the set of words on the alphabet X subject to the rule $pd = dp = \mathbf{1}$, where $\mathbf{1}$ denotes the empty word. Let W_0 be the subset of W made of words ending in the letter y . Therefore any word $w \in W_0$ can be written as

$$w = p^{k_1} y p^{k_2} y \cdots p^{k_n} y$$

for $k_1, \dots, k_n \in \mathbb{Z}$ using the identifications $p^{-1} = d$ and $p^0 = \mathbf{1}$. Let \mathcal{H} (respectively \mathcal{H}_0) denote the algebra $\mathcal{H} := \langle W \rangle_{\mathbb{Q}}$ (respectively $\mathcal{H}_0 := \langle W_0 \rangle_{\mathbb{Q}}$) spanned by the words in W (respectively W_0). For any $\lambda \in \mathbb{Q}^\times$ we define the bilinear product \sqcup_λ on \mathcal{H} by $\mathbf{1} \sqcup_\lambda w = w \sqcup_\lambda \mathbf{1} = w$ for any $w \in W$, and recursively for any words $u, v \in W$:

$$\begin{aligned}
yu \sqcup_\lambda v &:= u \sqcup_\lambda yv := y(u \sqcup_\lambda v), \\
pu \sqcup_\lambda pv &:= p(pu \sqcup_\lambda v + u \sqcup_\lambda pv + \lambda u \sqcup_\lambda v), \\
du \sqcup_\lambda dv &:= \frac{1}{\lambda} [d(u \sqcup_\lambda v) - u \sqcup_\lambda dv - du \sqcup_\lambda v], \\
du \sqcup_\lambda pv &:= pv \sqcup_\lambda du := d(u \sqcup_\lambda pv) - u \sqcup_\lambda v - \lambda du \sqcup_\lambda v.
\end{aligned}$$

Lemma 6.1. [5] *The pair $(\mathcal{H}, m_{\sqcup}^{(\lambda)})$ is a commutative, associative and unital algebra, and $(\mathcal{H}_0, m_{\sqcup}^{(\lambda)})$ is a subalgebra of $(\mathcal{H}, m_{\sqcup}^{(\lambda)})$.*

Next we define a *unital infinitesimal bialgebra* on \mathcal{H} . This will yield a coproduct $\overline{\Delta}$, which does not depend on λ . Then we will show that $(\mathcal{H}, m_{\sqcup}^{(\lambda)}, \overline{\Delta})$ is a proper bialgebra.

Recall that a unital infinitesimal bialgebra is a triple $(\mathcal{A}, m, \overline{\Delta})$ where (\mathcal{A}, m) is a unital associative algebra, and $\overline{\Delta} : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is coproduct with the following compatibility relation [17]:

$$(18) \quad \overline{\Delta}(u \cdot v) = (u \otimes \mathbf{1})\overline{\Delta}(v) + \overline{\Delta}(u)(\mathbf{1} \otimes v) - u \otimes v.$$

Our ansatz is as follows:

$$(19) \quad \overline{\Delta}(\mathbf{1}) := \mathbf{1} \otimes \mathbf{1},$$

$$(20) \quad \overline{\Delta}(p) := p \otimes \mathbf{1} + \mathbf{1} \otimes p,$$

$$(21) \quad \overline{\Delta}(y) := y \otimes \mathbf{1},$$

$$(22) \quad \overline{\Delta}(d) := 0.$$

This is motivated by the idea that $\overline{\Delta}$ is supposed to extend in some sense the deconcatenation coproduct to W . Here the letters p and py correspond to the letters x_0 and x_1 , respectively, which are primitives with respect to the usual deconcatenation coproduct.

Proposition 6.2. *The triple $(\mathcal{H}, m_{\square}^{(\lambda)}, \overline{\Delta})$ is a bialgebra, where $\overline{\Delta}: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is defined through (18) together with the initial values (19), (20), (21) and (22).*

Proof. First we have to prove that $\overline{\Delta}$ is well-defined. The initial values (19), (20) and (22) are consistent with the convention $dp = pd = \mathbf{1}$ since

$$\begin{aligned}\overline{\Delta}(dp) &= (d \otimes \mathbf{1})(\mathbf{1} \otimes p + p \otimes \mathbf{1}) - d \otimes p = \mathbf{1} \otimes \mathbf{1}, \\ \overline{\Delta}(pd) &= (\mathbf{1} \otimes p + p \otimes \mathbf{1})(\mathbf{1} \otimes d) - p \otimes d = \mathbf{1} \otimes \mathbf{1}.\end{aligned}$$

Next we show that for a given word w relation (18) is independent of the splitting point, i.e., for $w = w_1 w_2 = w'_1 w'_2$ we have

$$\overline{\Delta}(w_1 w_2) = \overline{\Delta}(w'_1 w'_2).$$

This is done by induction on the weight $\text{wt}(w)$. The base case is trivial. Now assume that $w = w_1 w_2 w_3$ with $\text{wt}(w_1)$, $\text{wt}(w_2)$, $\text{wt}(w_3)$, $\text{wt}(w_2 w_3)$, and $\text{wt}(w_1 w_2)$ all being smaller than $\text{wt}(w)$. Splitting w between w_1 and w_2 we observe

$$\begin{aligned}\overline{\Delta}(w_1 \cdot w_2 w_3) &= (w_1 \otimes \mathbf{1})\overline{\Delta}(w_2 w_3) + \overline{\Delta}(w_1)(\mathbf{1} \otimes w_2 w_3) - w_1 \otimes w_2 w_3 \\ &= (w_1 w_2 \otimes \mathbf{1})\overline{\Delta}(w_3) + (w_1 \otimes \mathbf{1})\overline{\Delta}(w_2)(\mathbf{1} \otimes w_3) - w_1 w_2 \otimes w_3 \\ &\quad + \overline{\Delta}(w_1)(\mathbf{1} \otimes w_2 w_3) - w_1 \otimes w_2 w_3\end{aligned}$$

and splitting w between w_2 and w_3 leads to

$$\begin{aligned}\overline{\Delta}(w_1 w_2 \cdot w_3) &= (w_1 w_2 \otimes \mathbf{1})\overline{\Delta}(w_3) + \overline{\Delta}(w_1 w_2)(\mathbf{1} \otimes w_3) - w_1 w_2 \otimes w_3 \\ &= (w_1 w_2 \otimes \mathbf{1})\overline{\Delta}(w_3) + (w_1 \otimes \mathbf{1})\overline{\Delta}(w_2)(\mathbf{1} \otimes w_3) + \overline{\Delta}(w_1)(\mathbf{1} \otimes w_2 w_3) \\ &\quad - w_1 \otimes w_2 w_3 - w_1 w_2 \otimes w_3,\end{aligned}$$

which coincide by induction hypothesis.

In the next step we prove coassociativity of $\overline{\Delta}$ by induction on the weight $\text{wt}(w)$. The base case is trivial. Now we have to distinguish three cases:

- Case 1: $w = yu$

$$\begin{aligned}((\overline{\Delta} \otimes \text{Id}) \circ \overline{\Delta})(yu) &= (\overline{\Delta} \otimes \text{Id})(y \otimes \mathbf{1})\overline{\Delta}(u) = \sum_{(u)} \overline{\Delta}(yu_1) \otimes u_2 \\ &= (y \otimes \mathbf{1} \otimes \mathbf{1})(\overline{\Delta} \otimes \text{Id})\overline{\Delta}(u) = (y \otimes \mathbf{1} \otimes \mathbf{1})(\text{Id} \otimes \overline{\Delta})\overline{\Delta}(u) \\ &= (\text{Id} \otimes \overline{\Delta})(y \otimes \mathbf{1})\overline{\Delta}(u) = ((\overline{\Delta} \otimes \text{Id}) \circ \overline{\Delta})(yu).\end{aligned}$$

- Case 2: $w = pu$

We have

$$\begin{aligned}((\overline{\Delta} \otimes \text{Id}) \circ \overline{\Delta})(pu) &= (\overline{\Delta} \otimes \text{Id})[(p \otimes \mathbf{1})\overline{\Delta}(u) + \mathbf{1} \otimes pu] \\ &= \sum_{(u)} \overline{\Delta}(pu_1) \otimes u_2 + \mathbf{1} \otimes \mathbf{1} \otimes pu \\ &= \sum_{(u)} [(p \otimes \mathbf{1})\overline{\Delta}(u_1) + \mathbf{1} \otimes pu_1] \otimes u_2 + \mathbf{1} \otimes \mathbf{1} \otimes pu \\ &= (p \otimes \mathbf{1} \otimes \mathbf{1})(\overline{\Delta} \otimes \text{Id})\overline{\Delta}(u) + \mathbf{1} \otimes ((p \otimes \mathbf{1})\overline{\Delta}(u)) + \mathbf{1} \otimes \mathbf{1} \otimes pu\end{aligned}$$

and

$$\begin{aligned}((\text{Id} \otimes \overline{\Delta}) \circ \overline{\Delta})(pu) &= (\text{Id} \otimes \overline{\Delta})[(p \otimes \mathbf{1})\overline{\Delta}(u) + \mathbf{1} \otimes pu] \\ &= \sum_{(u)} pu_1 \otimes \overline{\Delta}(u_2) + \mathbf{1} \otimes \overline{\Delta}(pu) \\ &= (p \otimes \mathbf{1} \otimes \mathbf{1})(\text{Id} \otimes \overline{\Delta})\overline{\Delta}(u) + \mathbf{1} \otimes ((p \otimes \mathbf{1})\overline{\Delta}(u)) + \mathbf{1} \otimes \mathbf{1} \otimes pu,\end{aligned}$$

which coincide by induction hypothesis.

- Case 3: $w = du$

$$\begin{aligned}
((\overline{\Delta} \otimes \text{Id}) \circ \overline{\Delta})(du) &= (\overline{\Delta} \otimes \text{Id}) [(d \otimes \mathbf{1})\overline{\Delta}(u) - d \otimes u] \\
&= \sum_{(u)} \overline{\Delta}(du_1) \otimes u_2 = \sum_{(u)} [(d \otimes \mathbf{1})\overline{\Delta}(u_1) \otimes u_2 - d \otimes u_1 \otimes u_2] \\
&= (d \otimes \mathbf{1} \otimes \mathbf{1})(\overline{\Delta} \otimes \text{Id})\overline{\Delta}(u) - d \otimes \overline{\Delta}(u) \\
&= (d \otimes \mathbf{1} \otimes \mathbf{1})(\text{Id} \otimes \overline{\Delta})\overline{\Delta}(u) - d \otimes \overline{\Delta}(u) \\
&= (\text{Id} \otimes \overline{\Delta}) [(d \otimes \mathbf{1})\overline{\Delta}(u) - d \otimes u] \\
&= ((\text{Id} \otimes \overline{\Delta}) \circ \overline{\Delta})(du).
\end{aligned}$$

We show $\overline{\Delta}(u) \sqcup_{\lambda} \overline{\Delta}(v) = \overline{\Delta}(u \sqcup_{\lambda} v)$ by induction on the sum of weights $s = \text{wt}(u) + \text{wt}(v)$, the case $s = 1$ being trivial.

- Case 1: $u = yu'$. Then we have:

$$\begin{aligned}
\overline{\Delta}(yu' \sqcup_{\lambda} v) &= \overline{\Delta}(y(u' \sqcup_{\lambda} v)) \\
&= \overline{\Delta}(y)(\mathbf{1} \otimes (u' \sqcup_{\lambda} v)) + (y \otimes \mathbf{1})\overline{\Delta}(u' \sqcup_{\lambda} v) - y \otimes (u' \sqcup_{\lambda} v) \\
&= (y \otimes \mathbf{1})\overline{\Delta}(u' \sqcup_{\lambda} v) \\
&= (y \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_{\lambda} \overline{\Delta}(v)) \\
&= ((y \otimes \mathbf{1})\overline{\Delta}(u')) \sqcup_{\lambda} \overline{\Delta}(v) \\
&= \overline{\Delta}(yu') \sqcup_{\lambda} \overline{\Delta}(v).
\end{aligned}$$

- Case 2: $u = du'$ and $v = dv'$. Two subcases occur: We compute:

$$\begin{aligned}
\overline{\Delta}(du' \sqcup_{\lambda} dv') &= \frac{1}{\lambda} \overline{\Delta}(d(u' \sqcup_{\lambda} v') - du' \sqcup_{\lambda} v' - u' \sqcup_{\lambda} dv') \\
&= \frac{1}{\lambda} \left[\overline{\Delta}(d)((\mathbf{1} \otimes (u' \sqcup_{\lambda} v')) + (d \otimes \mathbf{1})\overline{\Delta}(u' \sqcup_{\lambda} v') - d \otimes (u' \sqcup_{\lambda} v') \right. \\
&\quad \left. - \overline{\Delta}(du') \sqcup_{\lambda} \overline{\Delta}(v') - \overline{\Delta}(u') \sqcup_{\lambda} \overline{\Delta}(dv') \right] \\
&= \frac{1}{\lambda} \left[(d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_{\lambda} \overline{\Delta}(v')) - d \otimes (u' \sqcup_{\lambda} v') \right. \\
&\quad \left. - ((d \otimes \mathbf{1})\overline{\Delta}(u') - d \otimes u') \sqcup_{\lambda} \overline{\Delta}(v') - \overline{\Delta}(u') \sqcup_{\lambda} ((d \otimes \mathbf{1})\overline{\Delta}(v') - d \otimes v') \right] \\
&= \frac{1}{\lambda} \left[(d \otimes \mathbf{1})\overline{\Delta}(u') \sqcup_{\lambda} \overline{\Delta}(v') + \overline{\Delta}(u') \sqcup_{\lambda} (d \otimes \mathbf{1})\overline{\Delta}(v') + \lambda(d \otimes \mathbf{1})\overline{\Delta}(u') \sqcup_{\lambda} (d \otimes \mathbf{1})\overline{\Delta}(v') \right. \\
&\quad \left. - d \otimes (u' \sqcup_{\lambda} v') \right. \\
&\quad \left. - ((d \otimes \mathbf{1})\overline{\Delta}(u') - d \otimes u') \sqcup_{\lambda} \overline{\Delta}(v') - \overline{\Delta}(u') \sqcup_{\lambda} ((d \otimes \mathbf{1})\overline{\Delta}(v') - d \otimes v') \right] \\
&= \frac{1}{\lambda} \left[(d \otimes u') \sqcup_{\lambda} \overline{\Delta}(v') + \overline{\Delta}(u') \sqcup_{\lambda} (d \otimes v') - d \otimes (u' \sqcup_{\lambda} v') \right. \\
&\quad \left. + (d \otimes \mathbf{1})\overline{\Delta}(u') \sqcup_{\lambda} (d \otimes \mathbf{1})\overline{\Delta}(v') \right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\overline{\Delta}(du') \sqcup_{\lambda} \overline{\Delta}(dv') &= ((d \otimes \mathbf{1})\overline{\Delta}(u') - d \otimes u') \sqcup_{\lambda} ((d \otimes \mathbf{1})\overline{\Delta}(v') - d \otimes v') \\
&= (d \otimes \mathbf{1})\overline{\Delta}(u') \sqcup_{\lambda} (d \otimes \mathbf{1})\overline{\Delta}(v') - (d \otimes \mathbf{1})\overline{\Delta}(u') \sqcup_{\lambda} (d \otimes v') \\
&\quad - (d \otimes u') \sqcup_{\lambda} (d \otimes \mathbf{1})\overline{\Delta}(v') + (d \otimes u') \sqcup_{\lambda} (d \otimes v').
\end{aligned}$$

In view of $d \sqcup_\lambda d = -\frac{1}{\lambda}d$ and, more generally, $d \sqcup_\lambda du = -\frac{1}{\lambda}d \sqcup_\lambda u$ for any u , we get:

$$\begin{aligned}
& \overline{\Delta}(du' \sqcup_\lambda dv') - \overline{\Delta}(du') \sqcup_\lambda \overline{\Delta}(dv') \\
&= (d \otimes u') \sqcup_\lambda \left(\frac{1}{\lambda} - d \otimes \mathbf{1} \right) \overline{\Delta}(v') + \left(\frac{1}{\lambda} - d \otimes \mathbf{1} \right) \overline{\Delta}(u') \sqcup_\lambda (d \otimes v') \\
&= \sum_{(v')} \left(\frac{1}{\lambda} d \sqcup_\lambda v'_1 - d \sqcup_\lambda dv'_1 \right) \otimes (u' \sqcup_\lambda v'_2) + \sum_{(u')} \left(\frac{1}{\lambda} d \sqcup_\lambda u'_1 - d \sqcup_\lambda du'_1 \right) \otimes (u'_2 \sqcup_\lambda v') \\
&= 0.
\end{aligned}$$

- Case 3: $u = pu'$ and $v = pv'$. We observe

$$\begin{aligned}
\overline{\Delta}(pu') \sqcup_\lambda \overline{\Delta}(pv') &= \overline{\Delta}(p(u' \sqcup_\lambda pv' + pu' \sqcup_\lambda v' + \lambda u' \sqcup_\lambda v')) \\
&= (p \otimes \mathbf{1}) [\overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(pv') + \overline{\Delta}(pu') \sqcup_\lambda \overline{\Delta}(v') + \lambda \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v')] \\
&\quad + (\mathbf{1} \otimes p)(\mathbf{1} \otimes (u' \sqcup_\lambda pv' + pu' \sqcup_\lambda v' + \lambda u' \sqcup_\lambda v')) \\
&= (p \otimes \mathbf{1}) [\overline{\Delta}(u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v') + \overline{\Delta}(u') \sqcup_\lambda (\mathbf{1} \otimes pv')] \\
&\quad + ((p \otimes \mathbf{1}) \overline{\Delta}(u')) \sqcup_\lambda \overline{\Delta}(v') + (\mathbf{1} \otimes pu') \sqcup_\lambda \overline{\Delta}(v') + \lambda \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v')] \\
&\quad + \mathbf{1} \otimes (pu' \sqcup_\lambda pv')
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Delta}(pu') \sqcup_\lambda \overline{\Delta}(pv') &= ((p \otimes \mathbf{1}) \overline{\Delta}(u') + \mathbf{1} \otimes pu') \sqcup_\lambda ((p \otimes \mathbf{1}) \overline{\Delta}(v') + \mathbf{1} \otimes pv') \\
&= (p \otimes \mathbf{1}) \overline{\Delta}(u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v') + \mathbf{1} \otimes (pu' \sqcup_\lambda pv') \\
&\quad + (p \otimes \mathbf{1}) [\overline{\Delta}(u') \sqcup_\lambda (\mathbf{1} \otimes pv') + (\mathbf{1} \otimes pu') \sqcup_\lambda \overline{\Delta}(v')] \\
&= (p \otimes \mathbf{1}) [\overline{\Delta}(u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v') + (p \otimes \mathbf{1}) \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v') + \lambda \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v')] \\
&\quad + (p \otimes \mathbf{1}) [\overline{\Delta}(u') \sqcup_\lambda (\mathbf{1} \otimes pv') + (\mathbf{1} \otimes pu') \sqcup_\lambda \overline{\Delta}(v')] + \mathbf{1} \otimes (pu' \sqcup_\lambda pv')
\end{aligned}$$

- Case 4: $u = du'$ and $v = pv'$. We obtain

$$\begin{aligned}
\overline{\Delta}(du' \sqcup_\lambda pv') &= \overline{\Delta}(d(u' \sqcup_\lambda pv') - u' \sqcup_\lambda v' - \lambda du' \sqcup_\lambda v') \\
&= (d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(pv')) - \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v') - \lambda \overline{\Delta}(du') \sqcup_\lambda \overline{\Delta}(v') - d \otimes (u' \sqcup_\lambda pv') \\
&= (d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v')) + (d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_\lambda (\mathbf{1} \otimes pv')) \\
&\quad - \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v') - \lambda \overline{\Delta}(du') \sqcup_\lambda \overline{\Delta}(v') - d \otimes (u' \sqcup_\lambda pv') \\
&= (d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v')) + (d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_\lambda (\mathbf{1} \otimes pv')) - \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v') \\
&\quad - \lambda(d \otimes \mathbf{1}) \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v') + \lambda(d \otimes u') \sqcup_\lambda \overline{\Delta}(v') - d \otimes (u' \sqcup_\lambda pv')
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Delta}(du') \sqcup_\lambda \overline{\Delta}(pv') &= [(d \otimes \mathbf{1}) \overline{\Delta}(u') - d \otimes u'] \sqcup_\lambda [(p \otimes \mathbf{1}) \overline{\Delta}(v') + \mathbf{1} \otimes pv'] \\
&= (d \otimes \mathbf{1}) \overline{\Delta}(u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v') - (d \otimes \mathbf{1}) \overline{\Delta}(u') \sqcup_\lambda (\mathbf{1} \otimes pv') \\
&\quad - (d \otimes u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v') - d \otimes (u' \sqcup_\lambda pv') \\
&= (d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v')) - \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v') - \lambda(d \otimes \mathbf{1}) \overline{\Delta}(u') \sqcup_\lambda \overline{\Delta}(v') \\
&\quad - (d \otimes \mathbf{1})(\overline{\Delta}(u') \sqcup_\lambda (\mathbf{1} \otimes pv')) - (d \otimes u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v') - d \otimes (u' \sqcup_\lambda pv').
\end{aligned}$$

Using $(d \otimes u') \sqcup_\lambda (p \otimes \mathbf{1}) \overline{\Delta}(v') = -\lambda(d \otimes u') \sqcup_\lambda \overline{\Delta}(v')$ we conclude the proof. \square

Let $(\mathfrak{H}^1, m_*^{(-1)}, \Delta)$ be the quasi-shuffle Hopf algebra of weight -1 . Then \mathfrak{H}^0 is a left coideal of (\mathfrak{H}^1, Δ) and Corollary 3.2 implies that \mathfrak{H}^0 is a left coideal of $(\mathfrak{H}^{(-1)}, \Delta_\square)$.

Theorem 6.3. *Let $(\mathfrak{H}^1, m_*^{(-1)}, \Delta)$ be the quasi-shuffle Hopf algebra of weight -1 . The coproduct map $\Delta_{\square, op}$ restricted to the left coideal \mathfrak{H}^0 arising from the duality construction with respect to $\tilde{\tau}$, coincides with the coproduct $\bar{\Delta}$ restricted to \mathfrak{H}^0 defined by the infinitesimal bialgebra construction, i.e., the following diagram commutes:*

$$\begin{array}{ccc}
 \mathfrak{H}^0 & \xrightarrow{\bar{\Delta}} & \mathfrak{H}^{(-1)} \otimes \mathfrak{H}^0 \\
 \tilde{\tau} \downarrow & & \uparrow s \\
 & \mathfrak{H}^0 \otimes \mathfrak{H}^{(-1)} & \\
 & \uparrow \tilde{\tau} \otimes \tilde{\tau} & \\
 \mathfrak{H}^0 & \xrightarrow{\Delta} & \mathfrak{H}^0 \otimes \mathfrak{H}^1
 \end{array}$$

Proof. We prove the theorem by the length of the words. We have

$$\begin{aligned}
 \Delta_{\square, op}(p) &= s(\tilde{\tau} \otimes \tilde{\tau}) \Delta \tilde{\tau}(p) = s(\tilde{\tau} \otimes \tilde{\tau}) \Delta(y) \\
 &= s(\tilde{\tau} \otimes \tilde{\tau})(\mathbf{1} \otimes y + y \otimes \mathbf{1}) = \mathbf{1} \otimes p + p \otimes \mathbf{1} \\
 &= \bar{\Delta}(p).
 \end{aligned}$$

In the inductive step we distinguish two cases.

- First case: $w' = pw$ where w starts with a p . We observe

$$\begin{aligned}
 \bar{\Delta}(pw) &= (p \otimes \mathbf{1}) \bar{\Delta}(w) + \bar{\Delta}(p)(\mathbf{1} \otimes w) - p \otimes w \\
 &= (p \otimes \mathbf{1}) \Delta_{\square, op}(w) + \mathbf{1} \otimes pw \\
 &= (p \otimes \mathbf{1}) s(\tilde{\tau} \otimes \tilde{\tau}) \Delta(\tilde{\tau}(w)) + \mathbf{1} \otimes pw \\
 &= s(\tilde{\tau} \otimes \tilde{\tau}) [\Delta(\tilde{\tau}(w))(\mathbf{1} \otimes y) + \tilde{\tau}(pw) \otimes \mathbf{1}] \\
 &= s(\tilde{\tau} \otimes \tilde{\tau}) \Delta(\tilde{\tau}(pw)) \\
 &= \Delta_{\square, op}(pw).
 \end{aligned}$$

- Second case: $w' = p\tilde{w}$ where $\tilde{w} = y^k w$ ($k \in \mathbb{N}$) and w starts with p or is the empty word.

$$\begin{aligned}
 \bar{\Delta}(py^k w) &= (py^k \otimes \mathbf{1}) \bar{\Delta}(w) + \bar{\Delta}(py^k)(\mathbf{1} \otimes w) - py^k \otimes w \\
 &= (py^k \otimes \mathbf{1}) \Delta_{\square, op}(w) + \mathbf{1} \otimes py^k w \\
 &= (py^k \otimes \mathbf{1}) s(\tilde{\tau} \otimes \tilde{\tau}) \Delta(\tilde{\tau}(w)) + \mathbf{1} \otimes py^k w \\
 &= s(\tilde{\tau} \otimes \tilde{\tau}) [\Delta(\tilde{\tau}(w))(\mathbf{1} \otimes p^k y) + \tilde{\tau}(py^k w) \otimes \mathbf{1}] \\
 &= s(\tilde{\tau} \otimes \tilde{\tau}) \Delta(\tilde{\tau}(py^k w)) \\
 &= \Delta_{\square, op}(py^k w),
 \end{aligned}$$

using the fact that $\bar{\Delta}(py^k) = \mathbf{1} \otimes py^k + py^k \otimes \mathbf{1}$ ($k \in \mathbb{N}$) which is easily verified by induction. \square

6.2. The Schlesinger–Zudilin q -multiple zeta star vales. As we have shown in Theorem 5.12, the quasi-shuffle Hopf algebra of weight -1 , which corresponds to the quasi-shuffle product of Schlesinger–Zudilin q -MZSVs, induces a shuffle product for the OOOZ-model. In this section we show that we can do the reverse, i.e., we can use the quasi-shuffle-like product of the OOOZ-model to define a shuffle product for Schlesinger–Zudilin q -MZSVs.

In the first step, following [5], we introduce the quasi-shuffle product of the OOOZ-model. We define the operator $T: \mathfrak{H}^0 \rightarrow \mathfrak{H}^1$ by

$$T(z_m w) := z_m w - z_{m-1} w$$

for any $w \in \mathfrak{H}^0$ and $m \in \mathbb{N}$. Then we define the product $m_*^{\text{OOZ}}: \mathfrak{H}^0 \otimes \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$, $m_*^{\text{OOZ}}(u \otimes v) =: u *_{\text{OOZ}} v$, by

$$(QS1) \quad \mathbf{1} *_{\text{OOZ}} w := w *_{\text{OOZ}} \mathbf{1} := w;$$

$$(QS2) \quad z_m u *_{\text{OOZ}} z_n v := z_m(u *_1 T(z_n v)) + z_n(T(z_m u) *_1 v) + (z_{m+n} - z_{m+n-1})(u *_1 v)$$

for any $w \in \mathfrak{H}^0$, $u, v \in \mathfrak{H}^1$ and $m, n \in \mathbb{N}$, where $*_1$ was defined in Subsection 5.1.

Proposition 6.4 ([5]). *The map $\zeta^{\text{OOZ}}: (\mathfrak{H}^0, m_*^{\text{OOZ}}) \rightarrow \mathbb{Q}[[q]]$ is a character.*

Following [18] we review the shuffle product of classical multiple zeta star values. The shuffle product $\sqcup_\star: \mathbb{Q}\langle x_0, x_1 \rangle \otimes \mathbb{Q}\langle x_0, x_1 \rangle \rightarrow \mathbb{Q}\langle x_0, x_1 \rangle$ is defined iteratively by

- (i) $\mathbf{1} \sqcup_\star u = u \sqcup_\star \mathbf{1} = u$
- (ii) $au \sqcup_\star bv = a(u \sqcup_\star bv) + b(au \sqcup_\star v) - \delta(u)\tau(a)bv - \delta(v)\tau(b)au$

for any words $u, v \in \{x_0, x_1\}^*$ and letters $a, b \in \{x_0, x_1\}$, where δ is defined by

$$\delta(w) := \begin{cases} 1 & w = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

We apply Proposition 3.1 to the algebra $(\mathfrak{H}^0, m_\star^{\text{OOZ}})$ because for the coalgebra structure we have to go over to the completion of \mathfrak{H}^0 since the map T is not bijective.

We have the following result:

Theorem 6.5. *The product $m_\star^{\text{OOZ}}: \mathfrak{H}^0 \otimes \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$ induced by Proposition 3.1 defines a shuffle product for the Schlesinger–Zudilin q -MZSVs, which coincides with \sqcup_\star modulo terms of lower weight.*

Proof of Theorem 6.5. We perform two steps. First we need an alternative description of the shuffle product \sqcup_\star in terms of \sqcup :

Lemma 6.6. *The shuffle product $\sqcup_\star: \mathbb{Q}\langle x_0, x_1 \rangle \otimes \mathbb{Q}\langle x_0, x_1 \rangle \rightarrow \mathbb{Q}\langle x_0, x_1 \rangle$ can be calculated in terms of the ordinary shuffle product \sqcup via the formula*

$$ua \sqcup_\star vb = ua \sqcup vb - (u \sqcup v\tau(b))a - (u\tau(a) \sqcup v)b.$$

Proof. We define $ua \tilde{\sqcup} vb := ua \sqcup vb - (u \sqcup v\tau(b))a - (u\tau(a) \sqcup v)b$ and prove that $ua \tilde{\sqcup} vb = ua \sqcup_\star vb$ by induction over the sum of the weights of ua and vb . For the base case we observe

$$\begin{aligned} y \sqcup_\star y &= 2y^2 - 2xy = y \tilde{\sqcup} y, \\ y \sqcup_\star x &= yx + xy - x^2 - y^2 = y \tilde{\sqcup} x, \\ x \sqcup_\star x &= 2x^2 - 2yx = x \tilde{\sqcup} x. \end{aligned}$$

For the inductive step we have to consider two cases: In the first case we obtain

$$\begin{aligned} a_1ua_2 \sqcup_\star b &= a_1(ua_2 \sqcup_\star b) + ba_1ua_2 - \tau(b)a_1ua_2 \\ &= a_1(ua_2 \sqcup b) - a_1(u \sqcup \tau(b))a_2 - a_1u\tau(a_2)b + ba_1ua_2 - \tau(b)a_1ua_2 \\ &= a_1ua_2 \sqcup b - [a_1(u \sqcup \tau(b)) + \tau(b)a_1u]a_2 - a_1u\tau(a_2)b \\ &= a_1ua_2 \sqcup b - (a_1u \tilde{\sqcup} \tau(b))a_2 - a_1u\tau(a_2)b. \end{aligned}$$

Secondly, we observe that

$$\begin{aligned} a_1ua_2 \sqcup_\star b_1ub_2 &= a_1(ua_2 \sqcup_\star b_1ub_2) + b_1(a_1ua_2 \sqcup_\star ub_2) \\ &\quad - \delta(ua_2)\tau(a_1)b_1vb_2 - \delta(vb_2)\tau(v_1)a_1ua_2 \\ &= a_1(ua_2 \sqcup_\star b_1ub_2) + b_1(a_1ua_2 \sqcup_\star ub_2) \\ &= a_1(ua_2 \sqcup b_1v_1b_2) - a_1(u \sqcup b_1v\tau(b_2))a_2 - a_1(u\tau(a_2) \sqcup b_1v)b_2 \\ &\quad + b_1(a_1ua_2 \sqcup vb_2) - b_1(a_1u \sqcup v\tau(b_2))a_2 - b_1(a_1u\tau(a_2) \sqcup v)b_2 \\ &= a_1ua_2 \sqcup b_1v_1b_2 - [a_1(u\tau(a_2) \sqcup b_1v) + b_1(a_1u\tau(a_2) \sqcup v)]b_2 \\ &\quad - [a_1(u \sqcup b_1v\tau(b_2)) + b_1(a_1u \sqcup v\tau(b_2))]a_2 \\ &= a_1ua_2 \sqcup b_1v_1b_2 - (a_1u\tau(a_2) \sqcup b_1v)b_2 - (a_1u \sqcup b_1v\tau(b_2))a_2 \\ &= a_1ua_2 \tilde{\sqcup} b_1v_1b_2, \end{aligned}$$

which concludes the proof. \square

Now an explicit calculation shows that

$$\begin{aligned}
& up^a y^b \square_{\text{OOZ}} vp^c y^d \\
&= \tilde{\tau}(z_b z_0^{a-1} \tilde{\tau}(u) *_{\text{OOZ}} z_d z_0^{c-1} \tilde{\tau}(v)) \\
&= \tilde{\tau}(z_b z_0^{a-1} \tilde{\tau}(u) *_1 z_d z_0^{c-1} \tilde{\tau}(v) - z_b(z_0^{a-1} \tilde{\tau}(u) *_1 z_{d-1} z_0^{c-1} \tilde{\tau}(v)) \\
&\quad - z_d(z_{b-1} z_0^{a-1} \tilde{\tau}(u) *_1 z_0^{c-1} \tilde{\tau}(v)) - z_{b+d-1}(z_0^{a-1} \tilde{\tau}(u) *_1 z_0^{c-1} \tilde{\tau}(v))) \\
&= up^a y^b \square_1 vp^c y^d - (up^{a-1} \square_1 vp^c y^{d-1}) py^b - (up^a y^{b-1} \square_1 vp^{c-1}) py^d - (up^{a-1} \square_1 vp^{c-1}) py^{b+d-1}.
\end{aligned}$$

By Theorem 5.4 the product \square_1 coincides with \sqcup_1 which is the same as \sqcup modulo lower weight terms. This concludes the proof. \square

If we compare the definition of \sqcup_\star and Lemma 6.6 we see that the product \sqcup_\star can be calculated in terms of \sqcup_\star itself or in terms of the ordinary shuffle product \sqcup . Duality implies that this fact is also true for the quasi-shuffle product of the OOOZ-model.

Theorem 6.7. *Let $Y := \{z_k : k \in \mathbb{Z}\}$. The quasi-shuffle-like product $*_{\text{OOZ}} : \mathbb{Q}\langle Y \rangle \otimes \mathbb{Q}\langle Y \rangle \rightarrow \mathbb{Q}\langle Y \rangle$ is given by $\mathbf{1} *_{\text{OOZ}} w = w *_{\text{OOZ}} \mathbf{1} = w$ and*

$$\begin{aligned}
uz_m *_{\text{OOZ}} vz_n &= (u *_{\text{OOZ}} vz_n)z_m + (uz_m *_{\text{OOZ}} v)z_n + (u *_{\text{OOZ}} v)z_{n+m} - \delta(v)uz_m z_{n-1} \\
&\quad - \delta(u)vz_n z_{m-1} - \delta(v)uz_{n+m-1} - \delta(u)vz_{n+m-1} + \delta(u)\delta(v)z_{n+m-1}.
\end{aligned}$$

Proof. We define

$$\begin{aligned}
uz_m \tilde{*} vz_n &:= (u *_{\text{OOZ}} vz_n)z_m + (uz_m *_{\text{OOZ}} v)z_n + (u *_{\text{OOZ}} v)z_{n+m} - \delta(v)uz_m z_{n-1} \\
&\quad - \delta(u)vz_n z_{m-1} - \delta(v)uz_{n+m-1} - \delta(u)vz_{n+m-1} + \delta(u)\delta(v)z_{n+m-1}
\end{aligned}$$

and prove $w \tilde{*} w' = w *_{\text{OOZ}} w'$ by induction on the sum of the length of w and w' . The length one case is trivial. For length two we observe

$$z_m \tilde{*} z_n = z_m z_n + z_n z_m + z_{n+m} - z_m z_{n-1} - z_n z_{m-1} - z_{n+m-1} = z_m *_{\text{OOZ}} z_n.$$

Now we consider two cases. First of all, we have

$$\begin{aligned}
z_{a_1} uz_{a_2} \tilde{*} z_b &= (z_{a_1} u \tilde{*} z_b)z_{a_2} + z_{a_1} uz_{a_2} z_b + z_{a_1} uz_{a_2+b} - z_{a_1} uz_{a_2} z_{b-1} - z_{a_1} uz_{a_2+b-1} \\
&= z_{a_1}(u *_1 z_b)z_{a_2} + z_b z_{a_1} uz_{a_2} + z_{a_1+b} z_{a_2} \\
&\quad - z_{a_1}(u *_1 z_{b-1})z_{a_2} - z_b z_{a_1-1} uz_{a_2} - z_{a_1+b-1} uz_{a_2} \\
&\quad + z_{a_1} uz_{a_2} z_b + z_{a_1} uz_{a_2+b} - z_{a_1} uz_{a_2} z_{b-1} - z_{a_1} uz_{a_2+b-1} \\
&= z_{a_1}(uz_{a_2} *_1 z_b) + z_b z_{a_1} uz_{a_2} + z_{a_1+b} uz_{a_2} \\
&\quad - z_{a_1}(uz_{a_2} *_1 z_{b-1}) - z_b z_{a_1-1} uz_{a_2} - z_{a_1+b-1} uz_{a_2} \\
&= z_{a_1} uz_{a_2} *_{\text{OOZ}} z_b.
\end{aligned}$$

Furthermore we observe

$$\begin{aligned}
z_{a_1} uz_{a_2} \tilde{*} z_{b_1} vz_{b_2} &= (z_{a_1} u \tilde{*} z_{b_1} uz_{b_2})z_{a_2} + (z_{a_1} uz_{a_2} \tilde{*} z_{b_1} v)z_{b_2} + (z_{a_1} u \tilde{*} z_{b_1} v)z_{z_{a_2}+b_2} \\
&= z_{a_1}(u *_1 z_{b_1} vz_{b_2})z_{a_2} + z_{b_1}(z_{a_1} u *_1 vz_{b_1})z_{a_2} + z_{a_1+b_1}(u *_1 vz_{b_2})z_{a_2} \\
&\quad - z_{a_1}(u *_1 z_{b_1-1} vz_{b_2})z_{a_2} - z_{b_1}(z_{a_1-1} u *_1 vz_{b_1})z_{a_2} - z_{a_1+b_1-1}(u *_1 vz_{b_2})z_{a_2} \\
&\quad + z_{a_1}(uz_{a_2} *_1 z_{b_1} v)z_{b_2} + z_{b_1}(z_{a_1} uz_{a_2} *_1 v)z_{b_2} + z_{a_1+b_1}(uz_{a_1} *_1 v)z_{b_2} \\
&\quad - z_{a_1}(uz_{a_2} *_1 z_{b_1-1} v)z_{b_2} - z_{b_1}(z_{a_1-1} uz_{a_2} *_1 v)z_{b_2} - z_{a_1+b_1-1}(uz_{a_1} *_1 v)z_{b_2} \\
&\quad + z_{a_1}(u *_1 z_{b_1} v)z_{a_2+b_2} + z_{b_1}(z_{a_1} u *_1 v)z_{a_2+b_2} + z_{a_1+b_1}(u *_1 v)z_{a_2+b_2} \\
&\quad - z_{a_1}(u *_1 z_{b_1-1} v)z_{a_2+b_2} - z_{b_1}(z_{a_1-1} u *_1 v)z_{a_2+b_2} - z_{a_1+b_1-1}(u *_1 v)z_{a_2+b_2} \\
&= z_{a_1}(uz_{a_2} *_1 z_{b_1} vz_{b_2}) + z_{b_1}(z_{a_1} uz_{a_2} *_1 vz_{b_2}) + z_{a_1+b_1}(uz_{a_2} *_1 vz_{b_2}) \\
&\quad - z_{a_1}(uz_{a_2} *_1 z_{b_1-1} vz_{b_2}) - z_{b_1}(z_{a_1-1} uz_{a_2} *_1 vz_{b_2}) - z_{a_1+b_1-1}(uz_{a_2} *_1 vz_{b_2}) \\
&= z_{a_1} uz_{a_2} *_{\text{OOZ}} z_{b_1} vz_{b_2},
\end{aligned}$$

which concludes the proof. \square

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